On the $\frac{3}{4}$-Conjecture for Fix-Free Codes
A Survey

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Introduction

In this survey we concern ourself with the question, wether there exists a fix-free code for a given sequence of codeword lengths. For a given alphabet, we obtain the *Kraftsum* of a code, if we divide for every length the number of codewords of this length in the code by the total number of all possible words of this length and then take summation over all codeword lengths which appears in the code. The same way the Kraftsum of a lengths sequence $(l_1, \dots, l_n)$ is given by $\sum_{i=1}^{n} q^{-l_i}$, where q is the numbers of letters in the alphabet. Kraft and McMillan have shown in [1] (1956), that there exists a prefix-free code with codeword lengths of a certain lengths sequence, if the Kraftsum of the lengths sequence is smaller than or equal to one. Furthermore they have shown, that the converse also holds for all (uniquely decipherable) codes.[1] The question rises, if Kraft's and McMillan's result can be generalized to other types of codes? Throughout, we try to give an answer on this question for the class of fix-free codes. Since any code has Kraftsum smaller than or equal to one, this answers the question for the second implication of Kraft-McMillan's theorem. Therefore we pay attention mainly to the first implication.

A Kraft-McMillan inequality for fix-free codes

A *fix-free code* is a code, which is prefix-free and suffix-free, i.e. any codeword of a fix-free code is neither a prefix, nor a suffix of another codeword. Fix-free codes were first introduced by Schützenberg [3](1956) and Gilbert and Moore [4](1959), where they were called *never-self-synchronizing* codes. A good overview of fix-free code and some of their properties can be found for example in [11]. In the literature fix-free codes are also often called *affix-free*, *bifix-free* or *reversible-variable-length* (RCLs) codes.

Ahlswede, Balkenhol and Khachatrian propose in [5](1996) the conjecture that a Kraftsum of a lengths sequence smaller than or equal to $\frac{3}{4}$, imply the existence of a fix-free code with codeword lengths of the sequence. This is known as the $\frac{3}{4}$-conjecture for fix-free codes. Ahlswede, Balkenhol and Khachatrian give in [5] a justification of this conjecture. Especially they show that the conjecture holds for $\frac{1}{2}$ in place of $\frac{3}{4}$. Therefore a formulation of an existence theorem for fix-free codes in terms of a Kraftinequality similar to the first implication of Kraft-McMillan theorem is possible. Furthermore Ahlswede, Balkenhol and Khachatrian prove

[1] In this survey a code means a set of words, such that any message which is encoded with these words can be uniquely decoded. Therefore we omit in future the "uniquely decipherable" and write only "code".

in [5], that for any number γ bigger than $\frac{3}{4}$, there exists a lengths sequence with Kraftsum smaller than γ, for which no corresponding fix-free code exists. Otherwise, there are fix-free codes with Kraftsum bigger than $\frac{3}{4}$. For example the set of all words of fixed length n is a fix-free code with Kraftsum one. This shows that the first implication of Kraft-McMillans theorem can not hold for fix-free codes with Kraftsums bigger than $\frac{3}{4}$. Moreover a formulation of Kraft-McMillan theorem for fix-free codes, in such a way, that both implications hold for the same upper bound of the Kraftsum, is not possible. Originally Ahlswede, Balkenhol and Khachatrian examined only the case of a binary alphabet and a finite codes. However, Harada and Kobayashi generalized in [6](1999) all results of [5] for the case of q-ary alphabets and infinite codes.

Over the last years many attempts were done to prove the $\frac{3}{4}$-conjecture either for the general case of a q-ary alphabet or at least for the special case of a binary alphabet. All old results which are related to the $\frac{3}{4}$-conjecture can be found in [5]-[10]. Most of these results show that the conjecture holds for some special kinds of lengths sequences or that a weaker form of the conjecture is true. For example Harada and Kobayashi show in [6] the conjecture for two level codes, in the general case of q-ary alphabets or Yekhanin shows that the conjecture holds for $\frac{5}{8}$ in place of $\frac{3}{4}$ in the case of a binary alphabet. We survey in this survey all these old results about the $\frac{3}{4} - \textit{conjecture}$ and furthermore we obtain some new results, which are mostly generalizations of older results for the binary case to the case of a q-ary alphabet. A collection of all results can be found in the appendix at the end of this survey. Furthermore a small summary of this survey can be found at the end of this Introduction.

Applications of fix-free codes

A theorem which shows the existence of a fix-free code for given codeword lengths and a construction of fix-free codes for a given lengths sequence is quite important. Commonly variable length prefix-free codes are used for data compressing. However, fix-free codes have some properties which make them more favorable for a lot of applications compared with prefix-free codes. While fix-free codes are both prefix-free codes and suffix-free codes, it follows that they are bidirectionally decipherable, whereas prefix-free and suffix-free codes can be decoded only in one direction. A string which is encoded with a prefix-free code can instantaneously be decoded from the beginning toward the end, whereas a message, encoded by a suffix-free code, can be deciphered backwards, from the end to the beginning. Therefore the fix-free property ensures, that messages which are encoded with a fix-free code, can be read from both directions.

For example let $\mathcal{C}_1 := \{1, 00, 01\}, \mathcal{C}_2 := \{1, 10, 100\}$ and $C_3 := \{1, 00, 010\}$, then $\mathcal{C}_1$ is a prefix-free code, $\mathcal{C}_2$ is a suffix-free code and $\mathcal{C}_3$ is a fix-free code. We encode the letters N,I,A with the codes $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ respectively, as follows:

Source		$\mathcal{C}_1$	C_2	C_3
A	$\longrightarrow$	1	1	1
I	$\longrightarrow$	01	10	00
N	$\longrightarrow$	00	100	010

If the sequence 0001001 is a message which is encoded with $\mathcal{C}_1$, we can decipher the string step by step from left to right. The first codeword occurring in the string from the left hand side is 00. Since 00 is neither a prefix of 1 nor a prefix of 01, it follows that the message begins with an N. 01 is the next codeword of $\mathcal{C}_1$ which occurs from left to right in the string. While 01 is not a prefix of another codeword in $\mathcal{C}_1$, we obtain as the second letter I. If we proceed in this way, we decode the string 0001001 as the message NINA. However, if we try to read the string from right to left, we have some problems. The first codeword which occurs on the right hand side of the string is 1. This can mean, that the message ends with N or I, because 1 is a suffix of 01. If we proceed backward we obtain 01. This gives us the same problem, because it can mean, that the message ends with I or with NA. The next step backward gives us 001. This means obviously NA. However, this shows that the string 0001001 can not decoded codeword by codeword from right to left.

In the same way, a string which is encoded with $\mathcal{C}_2$ can be decoded step by step from the end toward the beginning, but it is in general not possible to decipher such a string by proceeding from left to right. For example, the string 100101001 is encoded with $\mathcal{C}_2$. It can be decoded as NINA, if we start at the end of the string, go backward to the beginning and decode directly every codeword when it occurs. If we start on the left hand side we have the same problem as above. Since 1 means, that the message begins with any letter. Since $\mathcal{C}_3$ is both prefix-free and suffix-free, we can decode a string which is generated by $\mathcal{C}_3$ from both sides. For example 010000101 can be read from the left-hand side as well as from the right-hand side as NINA.

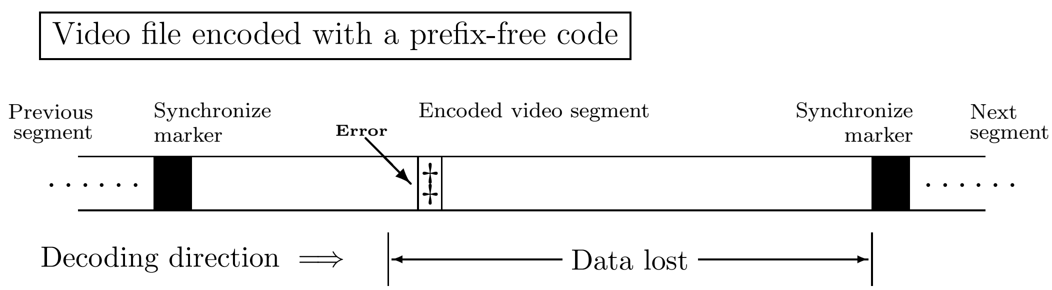
The bidirectional decoding property of fix-free codes is useful for many applications. For example, a string in a file which is compressed by a fix-free code, can be searched from both directions or a text which is encoded with a fix-free code can be decoded from both directions simultaneously. This reduce the decoding time to half, in comparison with decoding in one direction only.

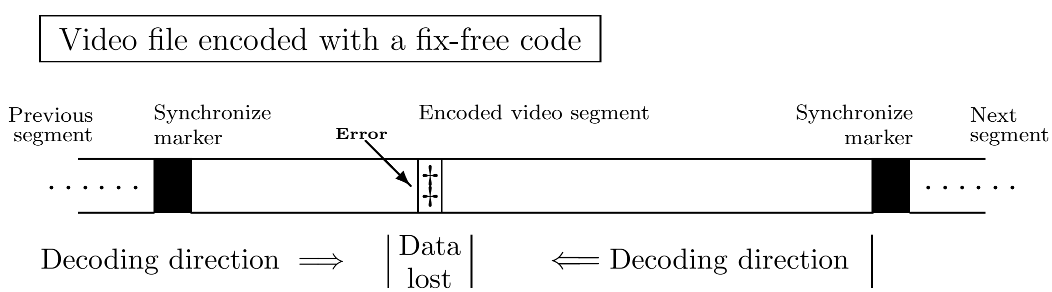
As another example: Suppose, that we have the problem to find a pattern $*P*$ in a given text which is encoded with some code. P is a string and $*$ represents an arbitrary string, maybe the empty string, which completes the string P to a word or a sentence respectively. If we want to complete the word or the sentence matched by P, we have to decode forward and backward from the position, where P was detected. We can do this, if all codewords have the same length. However, if we want to reduce the length of the encoded text, we have to use a variable length code. Since forward and backward encoding is necessary, the text has to be encoded with a fix-free code.

Related to the last example is the *Key Word In Context* (KWIC) display.(see Heaps [33]) A query for a text consists of one or several keywords and the location in the text where these words occur. This is done with a list of pointers for every keyword, which contains all positions of the appearance of the keyword in the text. A suitable way to present a query, is to show the context of the appearance of the keywords in the text. Therefore each of the k words in the text which appear before and behind the keywords are presented, where k is a fixed or a variable integer. This make bidirectional decoding necessary. If the wasteful way of encoding the text with a fixed-length code should be avoided, the text has to be encoded with a variable length fix-free code.

Another advantage of fix-free codes, in comparison with prefix-free codes, is their higher robustness in the presence of transmission errors. This is used for example in the development of video and media standards. Most parts of a video file are commonly encoded with a variable length prefix-free code (VCL), which minimize or reduce the average codeword length in comparison with a fixed-length code. Such a code is highly susceptible to transmission errors. There are two classes of bit errors which can occur, these are propagating errors and non-propagating errors. A non-propagating error gives only an incorrect decoding of the codeword in which the error occur. On the other hand a propagating error causes a loss of synchronization. In this case the bitstream behind the error will be decoded incorrectly or a decoding of the resisting bitstream is not possible. In some cases synchronizing will be reestablished later by itself, but also in this case often a lot of data is lost. Therefore commonly a frame of a video file is grouped into several segments. Each two of them are divided by a synchronization marker, such that a propagating error in one segment does not cause an erroneous decoding in another segment. Other kinds of error protection can be used to impose a more reliable code, if the data is transmitted trough a noisy channel. For example one can encode the video data with an error correcting code or with a comma-free code. Another method is to encode the most important parts of the video data with a more error robust code only. However, any of these more reliable coding schemes commonly increase the average codeword length.

This defeats the advantage of a careful use of resources, which is obtained by compressing the video data with a variable length code. Alternatively somebody can encode the video data with a fix-free code with the same or at least similar codeword lengths as the variable length prefix-free code. In this context a fix-free code is called a *reversible-variable-length-code* (RVLC). If an error burst occurs in a fix-free encoded segment, the decoder can jump to the synchronization marker at the end of the segment and decode backward to the error. Thus not all data in a segment behind an error is lost, if the video file is encoded with a fix-free code. This is shown in the pictures below. Furthermore it is sometimes possible to locate the position of an error in a segment by artificially causing additional errors and applying bidirectional decoding, where the results are compared with the initial decoder output.





An overview of error handling of fix-free codes and their applications in video encoding, especially in the video standards H.263 and MPEG-4, can be found in [13]-[17]. Furthermore in 1999 a data-partition structure based on reversible variable length codes (fix-free codes), has been adopted as the addition Annex V to the H.263++ video standard (see [19], [20] and also [17]).

The most important advantage of variable length codes in comparison with fixed-length codes, is their low average codeword length for a given source. A source is a set of finite symbols together with a probability distribution. For example, one can choose as a source the Latin alphabet together with the probability distribution which corresponds to the frequency of the Latin letters in a certain text or in a certain language. If the symbols in the alphabet are encoded by some code, the average codeword length is the sum of the codeword lengths weighted with the probabilities of the source. If we want to reduce decoding, encoding and transmission time or memory resources, it is favorable to choose a code with a low average codeword length. Therefore an optimal code, with respect to a source, is a code with minimal average codeword length. Huffman shows in [2] (1956) that it is possible for every source, to choose an optimal code which is prefix-free and that an optimal prefix-free code is also an optimal code. Furthermore he gave a construction of such prefix-free codes for a given source. Therefore optimal prefix-free codes are called *Huffman codes*.

Especially Huffman codes are *complete*, where finite complete codes are codes with Kraftsum one. It can also be said, that the code is a maximal code.[2] Since fix-free codes are especially prefix-free codes, the question rises, wether there exists a fix-free Huffman code for a given source. Fraenkel and Klein gave in [12] (1989) an algorithm which constructs a fix-free Huffman code for a given source, if there exists one. Furthermore the existence and properties of complete fix-free codes are studied extensively in [11].

On the other hand there exists sources, for which no fix-free Huffman codes exist. An example can be found in [7]. If $(0.7, 0.1, 0.1, 0.1)$ be the probability distribution of a source, $\{00, 01, 10, 11\}$ is the only complete fix-free code which corresponds to the source. The average codeword length of this code is 2, but $\{0, 11, 101, 1001\}$ is a fix-free code for the same source with average codeword length 1.6, where the Kraftsum is $\frac{15}{16}$. Since Huffman codes are complete codes, there does not exist fix-free Huffman code for the source $(0.7, 0.1, 0.1, 0.1)$. Therefore the question rises, how we can construct an optimal fix-free code for a given source. Although such an optimal fix-free code is not an optimal code in general, the examples above show that some applications make encoding with a fix-free code necessary or much more favorable than encoding with a prefix-free code. Since in general an optimal fix-free code is not complete, we have to pay attention to fix-free codes with Kraftsum smaller than one.

[2]Take in account that in general for infinite codes, completeness, maximality and to be code with Kraftsum one are not equivalent conditions.

First we could try to answer the question of the existence of a fix-free code for given lengths. If the $\frac{3}{4}$-conjecture holds, it would answer the question at least partially in an easy way. However, due to my knowledge it is not known, wether there are sources with an optimal fix-free code, which has Kraftsum smaller than or equal to $\frac{3}{4}$.

On the other hand a proof of the $\frac{3}{4}$-conjecture, will also give an upper bound for the average codeword length of an optimal fix-free code in the form of the noiseless coding theorem for prefix-free codes. If the probability distribution of a source is given by $P = (p_1, \ldots, p_n)$, the noiseless coding theorem states, that the average codeword length of a Huffman code for this source, is bounded by $H(P)$ from below and by $H(P)+1$ from above. Where $H(P)$ is the entropy of the source distribution, which is defined for binary codes as $H(P) = -\sum_{i=1}^{n} p_i \log_2 p_i$. While a fix-free code is also a prefix-free code, we have $H(P)$ also as a lower bound for the average codeword length of an optimal fix-free code. Ahlswede, Balkenhol and Khachatrian show in [5], that the conjecture holds for $\frac{1}{2}$ instead of $\frac{3}{4}$ and that this imply an upper bound of $H(P) + 2$ for the average codeword length of the optimal fix-free code. However, Yekhanin shows in [9] that the (binary) conjecture holds for $\frac{5}{8}$ in place of $\frac{3}{4}$ and this lowers the upper bound of an optimal fix-free code to $H(P)+4-\log_2 5$, which is approximately $H(P)+1.678$. However it can easily be shown, that the $\frac{3}{4}$-conjecture would improve this upper bound (for the binary case) to $H(P)+3+\log_2 3$, which is approximately $H(P)+1.415$. The proof of this and similar statements follows the same line as the proof of the original noiseless coding theorem for prefix-free codes, which can be found as an example in [21]. An upper bound for the average code word length of an optimal fix-free code can also be found in [7].

Another way to obtain "good" fix-free codes for a given source, is shown by Takishima, Wada and Murakami in [13](1995) and by Tsai and Wu in [15](2001). They gave there algorithms for construction of fix-free codes, which starts with the lengths of a Huffman code for a given source. This algorithms was improved by Laković and Villasenor in [14](2003). The average codeword length of the fix-free codes constructed by these algorithms for the English alphabet is shown in the tabular below.

Average codeword length for the English alphabet			
Huffman code	Takishima's fix-free code	Tsai's fix-free code	Laković's fix-free code
4.15572	4.36068	4.30678	4.25145

It was not proven, that the algorithms construct an optimal fix-free code for a given source and it seems to be, that they do not. However, we pay no more attention to this algorithms in this survey.

Summary of this survey

In this survey we focus mostly on results which shows the $\frac{3}{4}$-conjecture for special kinds of lengths sequences or on results which show that the conjecture holds in a weaker form. We distinguish between the conjecture for the binary case and the conjecture for the general q-ary case.

In Chapter 1 we give first an overview and a proof of the original Kraft-McMillan theorem for prefix-free codes. Then we give a justification of the $\frac{3}{4}$-conjecture for fix-free codes and examine different forms of the conjecture and the relations among themselves. Especially we show for the general q-ary case that the conjecture holds for $\frac{1}{2}$ in place of $\frac{3}{4}$ and that for every number bigger than $\frac{3}{4}$ the conjecture can not be hold. These theorems were first shown by Ahlswede, Balkenhol and Khachatrian in [5](1996) for the binary case. A generalization was shown by Harada and Kobayashi in [6](1999). Finally we study in Chapter 1 the existence of fix-free extensions of a fix-free code, i.e. we will see, that extensions of fix-free codes are crucially different to extensions of prfix-free codes.

Chapter 2 deals with the $\frac{3}{4}$-conjecture in the case of a q-ary alphabet. We prove three theorems which show that the conjecture holds for special kinds of lengths sequences. The first theorem occurs first for the binary case in [5](1996) and was generalized in [6](1999). It says, that the conjecture holds, if for two lengths of the sequence, there is a gap of at least twice time of the smaller length, where no other codeword length occur. The second theorem in the chapter shows that the conjecture holds for two level codes and it was proven by Harada and Kobayashi in [5]. Finally we show that the $\frac{3}{4}$-conjecture holds for finite sequences, if the numbers of codewords on each level is bounded by a term which depends on q and the smallest codeword length which occurs in the lengths sequence. This theorem was first shown by Kukorelly and Zeger in [10](2003) for the binary case. The generalization of this theorem in Chapter 2 to q-ary alphabets, is one of the new results in this survey.

Chapter 3 is a long preparation of Chapter 4. While we will construct fix-free codes from regular subgraphs in the de Bruijn digraph in Chapter 4, we give in Chapter 3 an introduction to the q-ary, n-th level de Bruijn digraph $\mathcal{B}_q(n)$. Especially we have to know the numbers of vertices, for which there exists a k-regular subgraph in $\mathcal{B}_q(n)$. De Bruijn graphs were introduced by de Bruijn [29](1946) and Good [30](1946) independently. After a small summary of some basic facts about digraphs and de Bruijn digraphs, we show that for every number L of vertices in $\mathcal{B}_q(n)$, there exists a cycle of length L in $\mathcal{B}_q(n)$. This was shown independently by Yoeli, Braynt, Heath , Killick, Golomb, Welch and Goldstein for binary de

Bruijn digraps. Lempel generalized this result to the q-ary de Bruijn digraphs. (see for all of these Lempel in [23](1971)). Especially cycles in $\mathcal{B}_q(n)$ are 1-regular subgraphs. Therefore we obtain, that there exist 1-regular subgraphs in $\mathcal{B}_q(n)$ for any possible number of vertices. At the end of the chapter, we try to answer the question of the existence of k-regular subgraphs in $\mathcal{B}_q(n)$ with certain numbers of vertices. We will see that there do not exist k-regular subgraphs in $\mathcal{B}_q(n)$ for vertices numbers smaller than k^n or for vertices numbers between k^n and $k^n - k^{n-1}$. Furthermore we give some constructions for k-regular subgraphs in $\mathcal{B}_q(n)$ with more than $k^n - k^{n-1}$ vertices. However, we will give no full answer on the question, for which numbers of vertices there are k-regular subgraphs in $\mathcal{B}_q(n)$.

In Chapter 4 we pay attention to a theorem which was claimed by Yekhanin in [8](2001). If the Kraftsum of the first level which occurs in a lengths sequence together with the Kraftsum of the following level is bigger than $\frac{1}{2}$, then from Yekanins theorem follows, that the $\frac{3}{4}$-conjecture holds. Yekanin claimed this theorem only for the binary case. However, no full proof of this theorem was published. Therefore we will give an own proof in Chapter 4, where we follow the proof idea which was proposed by Yekhanin in [8]. Furthermore we give a generalization of the theorem. For the proof of the theorem and its generalization, we introduce π-systems, which are special kinds of fix-free codes with Kraftsum $\left\lceil \frac{q}{2} \right\rceil q^{-1}$. Later we show, that π-systems can be extended to fix-free codes with Kraftsum smaller than or equal to $\frac{3}{4}$. This is called the π-system extension theorem, which we show in the first section of Chapter 4. In the second section of Chapter 4 we show, that π-systems with only two neighbouring levels and $L \cdot \left\lceil \frac{q}{2} \right\rceil$ codewords on the first level exist, if and only if there exists a $\left\lceil \frac{q}{2} \right\rceil$-regular subgraph of $\mathcal{B}_q(n)$ with L vertices. Furthermore we show that arbitrary one level π-systems exist. Since there exist cycles of arbitrary length in $\mathcal{B}_2(n)$, we obtain Yekhanins original theorem with the π-system extension theorem. However, in the generalization of Yekhanins theorem to the q-ary case, an extra condition for the existence of $\left\lceil \frac{q}{2} \right\rceil$-regular subgraph in $\mathcal{B}_q(n)$ occurs. Moreover we will show another version of all of these theorems, which uses other bounds than $\frac{3}{4}$ for the Kraftsum. To prove these more general versions, we work with k-regular subgraphs in $\mathcal{B}_q(n)$ instead of $\left\lceil \frac{q}{2} \right\rceil$-regular subgraphs in $\mathcal{B}_q(n)$. Mainly all of these results are new. Finally we prove in this chapter some minor new results for very special sequences by using the π-extension theorem for π-systems with more than two levels.

Chapter 5 is about the binary version of the $\frac{3}{4}$-conjecture. It begins with a summary of known results, which are shown only for the binary case. Then we give a simple construction of binary fix-free codes with the help of quaternary fix-free codes, by applying this construction to the results we have obtained in Chapter 2 and Chapter 4, we obtain some new results for the binary case of the $\frac{3}{4}$-conjecture. At the end of Chapter 5 we prove a result which was obtained by Yekanin in [9](2004), which shows, that the binary conjecture holds, if we replace in the conjecture $\frac{3}{4}$ by $\frac{5}{8}$. For this we use some special kinds of fix-free codes, for which the codewords with the same first letter and the same last letter are grouped in blocks. The blocks are ordered by the codeword lengths. Then we try to apply the technique of Yekhanins prove on the q-ary case. This gives us a new conjecture, which we prove for the ternary case. However, the new conjecture brings nothing new, because for all q bigger than 2 we obtain a Kraftsum smaller than $\frac{1}{2}$. Somebody might only be interested in the special block form of the fix-free codes, which occurs in the conjecture.

Finally the appendix contains all known old results and all new results of the survey, which are related to the $\frac{3}{4}$-conjecture.

A new result which is not contained in this survey

While this survey was in progress, K. Tichler has proven the conjecture which occurs in the last section of Chapter 5 for arbitrary q-ary alphabets. For a binary alphabet, the conjecture follows from Yekhanins proof in [9] of the $\frac{5}{8}$-version of the $\frac{3}{4}$-conjecture, which can also be found in the last section of Chapter 5. For a ternary alphabet the conjecture was first shown by the author of this survey, in the way as it is shown in Chapter 5. Some months after the author proposed the conjecture in Chapter 5, K. Tichler gave a counting proof, which shows that the conjecture holds for all q-ary alphabets. This conjecture gives no new results for the the $\frac{3}{4}$-conjecture for q-ary alphabets, because the fix-free codes in the conjecture have Kraftsums smaller than $\frac{1}{2}$ for $q > 2$ and the binary case was already shown by Yekhanin. However, somebody might be interested in the special block form of the fix-free codes which occurs in the conjecture. Furthermore K. Tichler has proven a variation of the conjecture in Chapter 5, which shows that for a ternary alphabet the $\frac{3}{4}$-conjecture holds for some $\gamma_3 > \frac{1}{2}$ in place of $\frac{3}{4}$. This is a new result for the $\frac{3}{4}$-conjecture in the case of ternary alphabets. Maybe such an variation of the conjecture in Chapter 5 is possible for all q. Since the proofs of K. Tichler are not worked out up to now, they won't be presented in this survey.

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Chapter 1

The Kraftinequality for fix-free codes

1.1 Notations and Definitions

Throughout this survey we denote with $\mathbb{N}$ the set of natural numbers without zero an with $\mathbb{N}_0$ the set of natural numbers with zero. If $\mathcal{M}$ is an arbitrary set, we write $\mathcal{P}(M)$ for the powerset of $\mathcal{M}$. This is the set which contains all subsets of $\mathcal{M}$ as its elements.

Let $\mathcal{A}$ be an arbitrary set, which we call an *alphabet* . The elements of $\mathcal{A}$ are called the *letters* of the alphabet $\mathcal{A}$. A *word of length n* over the alphabet $\mathcal{A}$ is a finite sequence of length n with values in $\mathcal{A}$. We write $a_1 \dots a_n \in \mathcal{A}^n$, for a finite sequence. The empty sequence is called the *empty word* or the *word of length 0* and is denoted by e.

For two words $w = w_1 \dots w_n \in \mathcal{A}^n$ and $v = v_1 \dots v_m \in \mathcal{A}^m$, we define the word $w \cdot v \in \mathcal{A}^{n+m}$ by the concatenation of the two sequences

$$w \cdot v := w_1 \dots w_n v_1 \dots v_m,$$

where we write wv in place of $w \cdot v$. Especially the operation $\cdot$ is associative and $we = ew = w$ for all $w \in \mathcal{A}^n$ and $n \in \mathbb{N}_0$.

We denote with $\mathcal{A}^*$ and $\mathcal{A}^+$ the set of all words on $\mathcal{A}$ with finite length and all finite words on $\mathcal{A}$ of length bigger than zero, respectively.

$$\mathcal{A}^* := \bigcup_{n=0}^{\infty} A^n \quad ; \quad \mathcal{A}^+ := \bigcup_{n=1}^{\infty} \mathcal{A}^n = \mathcal{A}^* - \{e\}$$

A *monoid* is a set $\mathcal{M}$ equipped with an associative binary operation $\cdot : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and a neutral element $e \in \mathcal{M}$. Obviously $(\mathcal{A}^*, \cdot, e)$ is a monoid. Let $(\mathcal{M}, \cdot, e)$ and $(\mathcal{N}, *, \mathbf{1})$ be two moniods. A map $\varphi : \mathcal{M} \to \mathcal{N}$ is called a *monoidhomomorphism*, if $\varphi(e) = \mathbf{1}$ and $\varphi(u \cdot v) = \varphi(u) * \varphi(v)$ for all $u, v \in \mathcal{M}$. The monoidhomomorphism φ is called an *monoidisomorphism*, if φ is a bijective map. In this case it follows that the inverse map φ^{-1} is also a monoidisomorphism.

For $v, w \in \mathcal{A}^*$ the word v is called a *prefix* of the word w, if there exists a word $u \in \mathcal{A}^*$ with $w = vu$. v is called a *suffix* of w, if $w = uv$ for some $u \in \mathcal{A}^*$.

A *factor* of w is a subword of w. This means $u \in \mathcal{A}^*$ is a factor of w, if there exists words $v, v' \in \mathcal{A}^*$ such that $w = vuv'$. i.e. e is a prefix, suffix and proper factor of any word in $\mathcal{A}^*$. Let $\mathcal{X} \subseteq \mathcal{A}^*$ and $w \in \mathcal{A}^*$. A *factorization* of w with words in $\mathcal{X}$, are words $x_1, \ldots, x_n \in \mathcal{X}$ such that

$$w = x_1 \ldots x_n .$$

For $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}^*$ and $v, w \in \mathcal{A}^*$ we define:

$$\begin{array}{lcl}
\mathcal{X}\mathcal{Y} & := & \{xy \in \mathcal{A}^* \,|\, x \in \mathcal{X},\, y \in \mathcal{Y}\}\,, \\
\mathcal{X}^{-1}\mathcal{Y} & := & \{z \in \mathcal{A}^* \,|\, \exists x \in \mathcal{X}\,,\, \exists y \in \mathcal{Y} \ \text{ with } \ y = xz\}\,, \\
\mathcal{X}\mathcal{Y}^{-1} & := & \{z \in \mathcal{A}^* \,|\, \exists x \in \mathcal{X}\,,\, \exists y \in \mathcal{Y} \ \text{ with } \ y = xz\}\,, \\
w^{-1}\mathcal{X} & := & \{w\}^{-1}\mathcal{X} \quad , \quad \mathcal{X}w^{-1} := \mathcal{X}\{w\}^{-1}\,, \\
w^{-1}v & := & \{w\}^{-1}\{v\} = \{u \in \mathcal{A}^* | v = wu\} \quad , \quad wv^{-1} := \{w\}\{v\}^{-1}\,, \\
w^n & := & \underbrace{w \ldots w}_{n\text{-times}} \ \text{ for } \ n \in \mathbb{N} \ \text{ and } \ w^0 := e\,, \\
\mathcal{A}^{-}\mathcal{X} & := & (\mathcal{A}^+)^{-1}\,\mathcal{X} \quad , \quad \mathcal{X}\mathcal{A}^{-} := \mathcal{X}\,(\mathcal{A}^+)^{-1}\,, \\
\mathcal{A}^{-n}\mathcal{X} & := & (\mathcal{A}^n)^{-1}\,\mathcal{X} \quad , \quad \mathcal{X}\mathcal{A}^{-n} := \mathcal{X}\,(\mathcal{A}^n)^{-1} \ \text{ for } \ n \in \mathbb{N}\,.
\end{array}$$

Especially $\mathcal{X}(\mathcal{X}^{-1}\mathcal{Y})$ is the set of all words in $\mathcal{Y}$ which have a prefix in $\mathcal{X}$, $(\mathcal{Y}\mathcal{X}^{-1})\mathcal{X}$ is the set of all words in $\mathcal{Y}$ which have a suffix in $\mathcal{X}$, $\mathcal{A}^{-}\mathcal{X}$ is the set of all proper suffixes of words in $\mathcal{X}$ and $\mathcal{X}\mathcal{A}^{-}$ is the set of all proper prefixes of words in $\mathcal{X}$. Furthermore we obtain for the sets $\mathcal{X}^n, \mathcal{X}^*, \mathcal{X}^+, \mathcal{X}^-\mathcal{Y}, \mathcal{Y}\mathcal{X}^-, X^{-n}\mathcal{Y}$ and $\mathcal{Y}\mathcal{X}^{-n}$:

$$\begin{array}{lcl}
\mathcal{X}^n & = & \{x_1 \ldots x_n \in \mathcal{A}^* \,|\, x_1, \ldots, x_n \in \mathcal{X}\}\ , \\
\mathcal{X}^* & = & \bigcup\limits_{n=0}^{\infty} \mathcal{X}^n \quad , \quad \mathcal{X}^+ = \bigcup\limits_{n=1}^{\infty} \mathcal{X}^n = \mathcal{X}^* - \{e\}\,, \\
\mathcal{X}^-\mathcal{Y} & = & (\mathcal{X}^+)^{-1}\,\mathcal{Y} \quad , \quad \mathcal{Y}\mathcal{X}^- = \mathcal{Y}\,(\mathcal{X}^+)^{-1}\,, \\
\mathcal{X}^{-n}\mathcal{Y} & = & (\mathcal{X}^n)^{-1}\,\mathcal{Y} \quad , \quad \mathcal{Y}\mathcal{X}^{-n} = \mathcal{Y}\,(\mathcal{X}^n)^{-1}\ .
\end{array}$$

It is easy to verify, that the following equations hold:

$$
\begin{array}{lcl}
\mathcal{X}^{-1}\mathcal{Y}\cap\mathcal{Y}=\emptyset & \Leftrightarrow & \mathcal{Y}\subseteq\mathcal{X}^{-1}\mathcal{Y}\Leftrightarrow e\in\mathcal{X}\,, \\
\mathcal{X}\mathcal{Y}^{-1}\cap\mathcal{X}=\emptyset & \Leftrightarrow & X\subseteq\mathcal{X}\mathcal{Y}^{-1}\Leftrightarrow e\in\mathcal{Y} \\
\mathcal{X}^{-1}\mathcal{Y}=\emptyset & \Leftrightarrow & \text{no word in } \mathcal{Y} \text{ has a prefix in } \mathcal{X}\,, \\
\mathcal{Y}\mathcal{X}^{-1}=\emptyset & \Leftrightarrow & \text{no word in } \mathcal{Y} \text{ has a suffix in } \mathcal{X}\,, \\
(\mathcal{X}\mathcal{Y})^{-1}\mathcal{Z} & = & \mathcal{Y}^{-1}(\mathcal{X}^{-1}\mathcal{Z}) =: \mathcal{X}^{-1}\mathcal{Y}^{-1}\mathcal{Z}\,, \\
\mathcal{X}(\mathcal{Y}\mathcal{Z})^{-1} & = & (\mathcal{X}\mathcal{Y}^{-1})\mathcal{Z}^{-1} =: \mathcal{X}\mathcal{Y}^{-1}\mathcal{Z}^{-1}\,, \\
(\mathcal{X}^{-1}\mathcal{Y})\mathcal{Z}^{-1} & = & (\mathcal{X}\mathcal{Y}^{-1})\mathcal{Z}^{-1} =: \mathcal{X}\mathcal{Y}^{-1}\mathcal{Z}^{-1}\,, \\
\mathcal{X}^{-1}(\mathcal{Y}\cup\mathcal{Z}) & = & \mathcal{X}^{-1}\mathcal{Y}\cup\mathcal{X}^{-1}\mathcal{Z}\,, \\
\mathcal{X}^{-1}(\mathcal{Y}\cap\mathcal{Z}) & = & \mathcal{X}^{-1}\mathcal{Y}\cap\mathcal{X}^{-1}\mathcal{Z}\,, \\
\mathcal{X}^{-1}(\mathcal{Y}-\mathcal{X}) & = & \mathcal{X}^{-1}\mathcal{Y}-\mathcal{X}^{-1}\mathcal{Z}\;.
\end{array}
$$

Similar equations holds for $(\mathcal{X}\cup\mathcal{Y})\mathcal{Z}^{-1}$, $(\mathcal{X}\cap\mathcal{Y})\mathcal{Z}^{-1}$ and $(\mathcal{X}-\mathcal{Y})\mathcal{Z}^{-1}$.

For $w\in\mathcal{A}^*$ and $a\in\mathcal{A}$ we denote with $|w|$ the length of the word w and with $<w|a>$ the number of occurrence of the letter a in w. For example let $\mathcal{A}=\{0,1\}$ and $w=11010$, then $|w|=5$, $<w|0>=2$ and $<w|1>=3$.

In the rest of this survey we suppose that alphabets are finite sets with at least two elements. Therefore let $|\mathcal{A}|=q$ for some $q\geq 2$.

Let $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers and $\mathcal{C}\subseteq\mathcal{A}^+$. We say the sequence $(\alpha_l)_{l\in\mathbb{N}}$ *fits to* the set $\mathcal{C}$ or $\mathcal{C}$ fits to $(\alpha_l)_{l\in\mathbb{N}}$, if $|\mathcal{C}\cap\mathcal{A}^l|=\alpha_l$ for all $l\in\mathbb{N}$.

For $\mathcal{C}\subseteq\mathcal{A}^*$ and $n\in\mathbb{N}_0$, the *Kraftsum* and the *n-th level Kraftsum* of $\mathcal{C}$ is defined as:

$$
S(\mathcal{C}) := \sum_{l=0}^{\infty}|\mathcal{C}\cap\mathcal{A}^l|q^{-l}\leq\infty \quad , \quad S_n(\mathcal{C}) := \sum_{l=0}^{n}|\mathcal{C}\cap\mathcal{A}^l|q^{-l}<\infty\,.
$$

A set $\mathcal{C}\subseteq\mathcal{A}^+$ is called a *code* on the alphabet $\mathcal{A}$, if every word in $\mathcal{C}^+$ has a unique factorization of words in $\mathcal{C}$.[1] This means for all $x\in\mathcal{C}^+$:

$$
\begin{array}{rl}
 & x=c_1\ldots c_n=d_1\ldots d_m \;\text{ with }\; c_i,d_j\in\mathcal{C}\;\forall\,1\leq i\leq n\,,\,1\leq j\leq m \\
\Rightarrow & n=m \;\text{ and }\; c_i=c_j\;\forall\,1\leq l\leq n=m
\end{array}
$$

The next proposition shows, that any message which is encoded with a code $\mathcal{C}\subseteq\mathcal{A}^+$, can be decoded uniquely.

[1] In information theoretical papers a set with the property above is commonly called an *unique decipherable* code and a code is an arbitrary subset of $\mathcal{A}^+$.

Proposition 1 *Let $\mathcal{C} \subseteq \mathcal{A}^+$ and $\mathcal{B}$ be another alphabet with $|\mathcal{B}| = |\mathcal{C}|$(, whereas $|\mathcal{B}| = \infty$ should be allowed).*[2] *Then $\mathcal{C}$ is a code if and only if there exists a bijection $\beta : \mathcal{B}^* \leftrightarrow \mathcal{C}^*$, such that $\beta(uv) = \beta(u)\beta(v)$ for all $u, v \in \mathcal{B}$.*

A proof of the proposition above, can be found for example in [11].

A code $\mathcal{C} \subseteq \mathcal{A}^+$ is called a *maximal code* , if for every word $c \in \mathcal{A}^+ - \mathcal{C}$ the set $\mathcal{C} \cup \{c\}$ is not a code. We call a set $\mathcal{D} \subseteq \mathcal{A}^+$ an *extension* of the code $\mathcal{C}$, if $\mathcal{D} \supseteq \mathcal{C}$ and $\mathcal{D}$ is a code. $\mathcal{D}$ is called a *maximal extension*, if $\mathcal{D}$ is a maximal code.

Let $\mathcal{M}$ be an arbitrary set. A binary relation $\preceq$ on $\mathcal{M}$ is called a *(partial) ordering,* if it is reflexive, antisymmetric and transetive. This means:

$$\begin{array}{lll} (1) & \forall\, a \in \mathcal{M} & : \quad a \preceq a\,, \\ (2) & \forall\, a, b \in \mathcal{M} & : \quad a \preceq b,\ b \preceq a \Rightarrow a = b\,, \\ (3) & \forall\, a, b, c \in \mathcal{M} & : \quad a \preceq b,\ b \preceq c \Rightarrow a \preceq c\,. \end{array}$$

We define $a \prec b$ as:

$$a \prec b \Leftrightarrow a \preceq b \ \text{ and } \ a \neq b\,.$$

The ordering $\preceq$ is called a *linear ordering,* if for all $a, b \in \mathcal{M}$ the elements a, b are comparable. This means $a \preceq b$ or $b \prec a$ for all $a, b \in \mathcal{M}$.

Let $\mathcal{C} \subseteq \mathcal{M}$. We call $a \in \mathcal{C}$ a *minimal element* of $\mathcal{C}$, if $b \not\preceq a$ for all $b \in \mathcal{C} - \{a\}$. We call a the *least element* of $\mathcal{C}$, if $a \preceq b$ for all $b \in \mathcal{C}$. If there exists a least element in $\mathcal{C}$, then it is also a minimal element of $\mathcal{C}$ and moreover, there do not exist other minimal elements in $\mathcal{C}$. In the same way we call $a \in \mathcal{C}$ a *maximal element* of $\mathcal{C}$, if $a \not\preceq b$ for all $b \in \mathcal{C} - \{a\}$ and a is called the *greatest element* of $\mathcal{C}$, if $b \preceq a$ for all $b \in \mathcal{C}$. If $\preceq$ is a linear ordering on $\mathcal{C}$, then every minimal element of $\mathcal{C}$ is the unique least element and every maximal element of $\mathcal{C}$ is the unique greatest element of $\mathcal{C}$.

An ordering $\preceq$ of a set $\mathcal{M}$ is called a *well-ordering,* if it is a linear ordering and if every nonempty subset of $\mathcal{M}$ has a least element.

Let $\preceq_1$ be an ordering of a set $\mathcal{M}_1$ and $\preceq_2$ be an ordering of a set $\mathcal{M}_2$. We call the orderings $(\mathcal{M}_1, \preceq_1)$ and $(\mathcal{M}_2, \preceq_2)$ *isomorph,* if there exists a bijection $\varphi : \mathcal{M}_1 \leftrightarrow \mathcal{M}_2$ such that $a \preceq_1 b$ if and only if $\varphi(a) \preceq_2 \varphi(b)$ for all $a, b \in \mathcal{M}_1$. We call φ an *isomorphism* between $(\mathcal{M}_1, \preceq_1)$ and $(\mathcal{M}_2, \preceq_2)$.

[2] Since $\mathcal{A}$ is finite, the set $\mathcal{B}$ is at most countable.

Let us give two examples:

Example 1 Let $\mathcal{M}$ be an arbitrary set. The subsetrelation $\subseteq$ is a ordering of $\mathcal{P}(\mathcal{M})$, with greatest element $\mathcal{M}$ and least element $\emptyset$.

Example 2 Let $n \in \mathbb{N}$, then $0 < 1 < 2 < \ldots < n-1$ is a well-ordering of $\{0, \ldots, n-1\}$. Furthermore any linear ordering of a set $\mathcal{M}$ with $|\mathcal{M}| = n$ is isomorphically to this ordering. The ordering $0 < 1 < 2 < \ldots$ of $\mathbb{N}_0$ is also a well-ordering. This ordering of the natural numbers is denoted by ω. However, there exist much more well-orderings and linear orderings of $\mathbb{N}_0$ which are not isomorphically to ω. Let us define for example $1 \prec 2 \prec 3 \prec \ldots$ and $n \prec 0$ for all $n \in \mathbb{N}$. Then this is a well-ordering of $\mathbb{N}_0$ which is not isomorphic to ω.[3]

Let $\preceq$ be an ordering of a set $\mathcal{M}$. A subset $\mathcal{C} \subseteq \mathcal{M}$ is called a *chain*, if $\mathcal{C}$ is linear ordered by $\preceq$. $\mathcal{C}$ is called an *antichain* if all elements of $\mathcal{C}$ are incomparable. This means $a \not\preceq b$ and $b \not\preceq a$ for all $a, b \in \mathcal{C}$ with $a \neq b$.

We call an element $a \in \mathcal{M}$ a *lower bound* of $\mathcal{C}$ and an element $b \in \mathcal{M}$ an *upper bound* of $\mathcal{C}$, if $a \preceq c$ and $c \preceq b$ for all $c \in \mathcal{C}$. Obviously $a, b \in \mathcal{C}$ if and only if a is the least element of $\mathcal{C}$ and b is the greatest element of $\mathcal{C}$.

The next lemma is known as Zorn's lemma, which can be found in most books about set theory (for example [26]), therefore we omit a proof.

Lemma 2 (Zorn's lemma) *Let $\preceq$ be an ordering of a set $\mathcal{M}$. If every chain has an upper bound, then there exists a maximal element in $\mathcal{M}$.*

It is well known in set theory, that Zorn's lemma is an equivalence of the Axiom of Choice (see for example [26]). Therefore proofs which use Zorn's lemma are none-constructive proofs. As an example for Zorn's lemma we prove, that any code has a maximal extension.

Proposition 3 *Let $|\mathcal{A}| = q \geq 2$ and $\mathcal{C} \subseteq \mathcal{A}^+$ be a code. There exists a maximal code $\mathcal{D}$ with $\mathcal{C} \subseteq \ D \subseteq \mathcal{A}^+$.*

Proof: Let $\mathcal{C} \subseteq \mathcal{A}^+$ be a code. We define $\mathcal{M} \subseteq \mathcal{P}(\mathcal{A}^+)$ as the set of codes extensions of $\mathcal{C}$.

$$\mathcal{M} := \{\mathcal{D} \subseteq \mathcal{A}^* \mid \mathcal{C} \subseteq \mathcal{Y} \text{ and } \mathcal{Y} \text{ is a code }\}$$

[3] In general every well-ordering is isomorphically to the ordertype of a unique ordinal number.

Obviously $\mathcal{M}$ is ordered by $\subseteq$ and any maximal element of $\mathcal{M}$ is a maximal extension of $\mathcal{C}$. Therefore it is sufficient to show, that $\mathcal{M}$ has at least one maximal element.

Let $\mathcal{M}_c \subseteq \mathcal{M}$ be a chain in $\mathcal{M}$ and $\mathcal{D}' := \bigcup_{\mathcal{D} \in \mathcal{M}_c} \mathcal{D}$. Then $\mathcal{C} \subseteq \mathcal{D}'$ and $\mathcal{D} \subseteq \mathcal{D}'$ for all $\mathcal{D} \in \mathcal{M}_c$. Let us assume that $\mathcal{D}'$ is not a code. Then there exists words $c_1, \ldots, c_{n+m} \in \mathcal{D}'$ such that:

$$c_1 \ldots c_n = c_{n+1} \ldots c_{n+m}. \tag{1.1}$$

Each of the c_i's is contained in a code $\mathcal{D}_i \in \mathcal{M}_m$. Since $\mathcal{M}_c$ is linear ordered by $\subseteq$, it follows that there exists $j \in \{1, \ldots, n+m\}$ with $\mathcal{D}_i \subseteq \mathcal{D}_j$ for all $1 \leq i \leq n+m$. We obtain that $c_1, \ldots, c_{n+m} \in \mathcal{D}_j$. This is a contradiction, because $\mathcal{D}_j$ is a code. Thus (1.1) can not hold. This shows that $\mathcal{D}'$ is a code, i.e. $\mathcal{D}' \in \mathcal{M}$. Therefore $\mathcal{D}'$ is an upper bound of $\mathcal{M}_c$ in $\mathcal{M}$. By Zorn's lemma follows, that $\mathcal{M}$ has a maximal element. **q.e.d**

Let $\preceq$ be an ordering of a set $\mathcal{T}$. We call $(\mathcal{T}, \preceq)$ a *tree* , if for every $a \in \mathcal{T}$ the set $\{b \in \mathcal{T} | b \prec a\}$ is well-ordered by $\preceq$ and if $\mathcal{T}$ has a least element, which is called the *root* of the tree $\mathcal{T}$. For a tree $\mathcal{T}$, any chain in $\mathcal{T}$ is well-ordered by $\preceq$.

An element $a \in \mathcal{T}$ is called a *node* of the tree. Furthermore it is called a *finite node on the $l(a)$-th level*, if $|\{b \in \mathcal{T} | b \prec a\}| = l(a) < \infty$. If $a \in \mathcal{T}$ is a finite node, then the chain $\{b \in \mathcal{T} | b \prec a\}$ is isomorphic to the well-ordering $0 < 1 < 2 < \ldots < l(a) - 1$.

Let $l \in \mathbb{N}_0$. The l-th level of the tree is defined as the set $\mathcal{T}(l) := \{a \in \mathcal{T} | l(a) = l\}$. For any $a, b \in \mathcal{T}(l)$ with $a \neq b$ the nodes a and b are incomparable. If $\mathcal{T}(n) = \emptyset$, then $\mathcal{T}(l) = \emptyset$ for all $l \geq m$.

We call the tree $\mathcal{T}$ has *height* h for some $h \in \mathbb{N}$, if $\mathcal{T}(h-1) \neq \emptyset$ and $\mathcal{T}(l) = \emptyset$ for all $l \geq h$. This means the heights of $\mathcal{T}$ is the smallest level, which is empty. We write $\mathcal{T}$ has height ω or $\mathcal{T}$ is an *ω-tree,* if all nodes of $\mathcal{T}$ are finite and $\mathcal{T}(l) \neq \emptyset$ for all $l \in \mathbb{N}_0$. If $\mathcal{T}$ is an ω-tree, then any chain in $\mathcal{T}$ is either isomorphic to $0 < 1 < \ldots < n$ for some $n \in \mathbb{N}_0$ or it is isomorphic to ω.[4]

A *branch* of a tree $\mathcal{T}$ is a maximal chain $\mathcal{C}$ in $\mathcal{T}$. This means $\mathcal{C}$ is a chain and for every $a \in \mathcal{T} - \mathcal{C}$ the set $\mathcal{C} \cup \{a\}$ is not a chain. Let $\mathcal{T}$ be a tree with finite heights or an ω-tree and let $\mathcal{C}$ be a branch in $\mathcal{T}$. We call $l \in \mathbb{N}$ the length of

[4] For an arbitrary tree any chain is isomorphic to the ordertype of a (unique) ordinal number and the ordinal number which is isomorphic to $\{b \in \mathcal{T} | b \prec a\}$ is the level of a. The height of $\mathcal{T}$ is the smallest empty level. Furthermore for any chain the corresponding ordinal is smaller tan or equal to the heights of the tree.

the branch, if $\mathcal{C}$ is isomorphic to $0 < 1 < 2 \ldots < l-1$ and we call $\mathcal{C}$ a branch of length ω, if $\mathcal{C}$ is isomorphic to ω. If $\mathcal{T}$ has height $h \in \mathbb{N}$, then there exists a branch of length h.

The next lemma shows, that this holds also for ω-trees which have finite levels.

Lemma 4 (König's lemma) *Let $\mathcal{T}$ be an ω-tree. If any level contains a finite number of nodes, then there exists a branch of length ω in $\mathcal{T}$.*

The lemma doesn't hold if the levels of $\mathcal{T}$ contain infinite nodes. An example of such a ω-tree and a proof of the lemma can be found in [26]. In the proof of the lemma the Axiom of Choice is used, but the lemma is not an equivalent of the Axiom of Choice. However, just as the Axiom of choice, Königs lemma can not be proven or disproven with the set axioms of Zermalo-Fränklel. This can be found for example in [27]. Therefore also proofs which use Königs's lemma are none-constructive.

Let $\mathcal{A}$ be an arbitrary set. For $x, y \in \mathcal{A}^*$ we define $x \overset{p}{\preceq} y$ if x is a prefix of y and $x \overset{s}{\preceq} y$ if x is a suffix of y. It is easy to verify that $(\mathcal{A}^*, \overset{p}{\preceq})$ and $(\mathcal{A}^*, \overset{s}{\preceq})$ are both ω-trees with root e, which we call the prefix-tree and the suffix-tree, respectively. Furthermore the l-th level of both trees is given by $\mathcal{A}^l$ for all $l \in \mathbb{N}_0$.

1.2 Fix-free codes

A set $\mathcal{C} \subseteq \mathcal{A}^*$ is called *prefix-free*, if no word in $\mathcal{C}$ is a prefix of another word in $\mathcal{C}$ and it is called *suffix-free*, if no word in $\mathcal{C}$ is a suffix of another word in $\mathcal{C}$. The set $\mathcal{C}$ is called *fix-free* or *bifix-free* , if it is prefix- and suffix-free. Since e is a prefix and a suffix of every word in $\mathcal{A}^*$, we obtain, that $\{e\}$ is the only prefix- suffix- and fix-free set, which contains e as an element. Therefore we obtain:

$$\begin{array}{lcl} \mathcal{C} \text{ is prefix-free} & \Leftrightarrow & \mathcal{C}\mathcal{A}^+ \cap \mathcal{C} = \emptyset\,, \\ \mathcal{C} \text{ is suffix-free} & \Leftrightarrow & \mathcal{A}^+\mathcal{C} \cap \mathcal{C} = \emptyset\,. \end{array}$$

The set $\mathcal{C}$ is called *fix-free* or *bifix-free* , if it is prefix- and suffix-free.

For an arbitrary set $\mathcal{C} \subseteq \mathcal{A}^*$ the *prefix-, suffix- and bifix-shadow* of $\mathcal{C}$ on the n-th level are defined as:

$$\begin{array}{lcccl} \Delta_P^n(\mathcal{C}) & := & \bigcup\limits_{l=0}^{n} (\mathcal{C} \cap \mathcal{A}^l)\mathcal{A}^{n-l} & \subseteq & \mathcal{A}^n\,, \\ \Delta_S^n(\mathcal{C}) & := & \bigcup\limits_{l=0}^{n} \mathcal{A}^{n-l}(\mathcal{C} \cap \mathcal{A}^l) & \subseteq & \mathcal{A}^n\,, \\ \Delta_B^n(\mathcal{C}) & := & \Delta_P^n(\mathcal{C}) \cup \Delta_S^n(\mathcal{C}) & \subseteq & \mathcal{A}^n\,. \end{array}$$

Proposition 5 *Every subset of $\mathcal{A}^*$ which is prefix- suffix- or fix-free and not equal to $\{e\}$ is also a code.*

Proof: Let $\mathcal{C} \subseteq \mathcal{A}^+$ be a prefix-free set and let $\mathbf{w} \in \mathcal{C}^+$, $x_1, \ldots, x_n, y_1, \ldots y_m \in \mathcal{C}$, such that $\mathbf{w} = x_1 \ldots x_n = y_1 \ldots y_m$, where $n \leq m$. Let us assume that there exists i with $x_i \neq y_i$. If we choose i minimal, we obtain, that either x_i is a prefix of y_i or y_i is a prefix of x_i. This is a contradiction, because $\mathcal{C}$ is prefix-free. Therefore $x_i = y_i$ for all $1 \leq i \leq n$. Furthermore from $x_1 \ldots x_n = y_1 \ldots y_m$ and $e \notin \mathcal{C}$ follows that $n = m$. This shows that $\mathcal{C}$ is a code. The proof for suffix-free sets follows the same steps. **q.e.d**

The next proposition shows how we obtain a prefix-free code from an arbitrary set $\mathcal{X} \subseteq \mathcal{A}^+$.

Proposition 6 *Let $\mathcal{X} \subseteq \mathcal{A}^+$ with $\mathcal{X} \neq \emptyset$ and $\mathcal{Y} := \mathcal{X} - \mathcal{X}\mathcal{A}^+$. Then $\mathcal{Y} \neq \emptyset$ is a prefix-free code and $\mathcal{X}\mathcal{A}^* = \mathcal{Y}\mathcal{A}^*$.*

Proof: Let $x, z \in \mathcal{X}$, such that $|x| < |z|$ for all $z \in \mathcal{X}$, then $x \notin \mathcal{X}\mathcal{A}^+$. Hence $\mathcal{X} \neq \emptyset$. From $\mathcal{X} \subseteq \mathcal{Y}$ follows $\mathcal{Y}\mathcal{A}^+ \subseteq \mathcal{X}\mathcal{A}^+$. Since $\mathcal{Y} = \mathcal{X} - \mathcal{X}\mathcal{A}^+$, we obtain

$\mathcal{Y} \cap \mathcal{Y}\mathcal{A}^+ \subseteq \mathcal{Y} \cap \mathcal{X}\mathcal{A}^+ = \emptyset$. This shows that $\mathcal{Y}$ is a prefix-free code, because $e \notin \mathcal{X}$.

Obviously $\mathcal{Y}\mathcal{A}^* \subseteq \mathcal{X}\mathcal{A}^*$ holds. We show the other direction. Let $x \in \mathcal{X}$. If $x \in \mathcal{Y}$, then $x \in \mathcal{Y}\mathcal{A}^*$. Otherwise $x \in \mathcal{X}\mathcal{A}^+$, whence $x = x_1 z_1$ for some $x_1 \in \mathcal{X}, z \in \mathcal{A}^+$. Especially we obtain $|x_1| < |x|$. By induction on the length of x it follows, that $x = x_n z_1 \dots z_n$ for some $z_1, \dots, z_n \in \mathcal{A}^+$ and $x_n \in \mathcal{Y}$. It follows that $x \in \mathcal{Y}\mathcal{A}^+$. Therefore we obtain $\mathcal{X} \subseteq \mathcal{Y}\mathcal{A}^*$. Since $\mathcal{X}\mathcal{A}^* \subseteq \mathcal{Y}\mathcal{A}^*\mathcal{A}^* = \mathcal{Y}\mathcal{A}^*$, this shows $\mathcal{X}\mathcal{A}^* = \mathcal{Y}\mathcal{A}^*$. **q.e.d**

A prefix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ is called a *maximal prefix-free code* if for every $c \in \mathcal{A}^+ - \mathcal{C}$ the set $\mathcal{C} \cup \{c\}$ is not prefix-free. In the same way *maximal suffix-free codes* and *maximal fix-free codes* are defined. The question rises, if a maximal fix-free code is also a maximal code? Indeed in general, this is not the case for infinite codes. However, it is true for finite codes.

A set $\mathcal{X} \subseteq \mathcal{A}^*$ is called *dense*, if

$$\mathcal{A}^* w \mathcal{A}^* \cap \mathcal{X} \neq \emptyset \quad \text{for all} \quad w \in \mathcal{A}^* .$$

The set $\mathcal{X}$ is called *thin*, if $\mathcal{X}$ is not a dense set. This means $\mathcal{X}$ is a thin set if and only if there exists a word $w \in \mathcal{A}^*$ such that $\mathcal{A}^* w \mathcal{A}^* \cap \mathcal{X} = \emptyset$.

Proposition 7 *Every finite set is a thin set, as well.*

Proof: Let $\mathcal{X} \subseteq \mathcal{A}^*$ be a finite set. Since $\mathcal{X}$ is finite, there exists a word $w \in \mathcal{A}^*$ such that $|w| > |x|$ for all $x \in \mathcal{X}$. It follows that $\mathcal{A}^* w \mathcal{A}^* \cap \mathcal{X} = \emptyset$. **q.e.d**

The next two examples shows, that dense fix-free codes and infinite thin fix-free codes exist.

Example 3 Let $\mathcal{A} := \{0, 1\}$ and

$$\mathcal{D} := \left\{ w \in \mathcal{A}^+ \,\middle|\, < w|0 >=< w|1 > \quad \text{and} \quad < u|0 > \neq < u|1 > \quad \text{for all} \quad u \in w\mathcal{A}^- \right\} .$$

This means, if $w \in \mathcal{D}$, then the number of 0's in the word w is equal to the number of 1's in the word, but $< u|0 > \neq < u|1 >$ for any proper prefix u of w. Obviously $\mathcal{D}$ is a prefix-free code. Let us assume that there exists $w, w' \in \mathcal{D}$ such that w' is a proper suffix of w. Then there exists a word $u \in \mathcal{A}^+$ with $w = uw'$. We obtain:

$$\begin{aligned} < w|0 > &= < u|0 > + < w'|0 >=< u|0 > + < w'|1 > \\ &\neq < u|1 > + < w'|1 >=< w|1 > . \end{aligned}$$

This is a contradiction. Therefore $\mathcal{D}$ is a fix-free code. $\mathcal{D}$ is called the *binary Dyckcode.*

Let $w \in \mathcal{A}^+$, then $0^{2<w|1>}w1^{|w|} \in \mathcal{A}^* w \mathcal{A}^*$. We obtain:

$$\begin{aligned} \left\langle 0^{2<w|1>}w1^{|w|}\,|0\right\rangle &= 2<w|1>+<w|0>=|w|+<w|1> \\ &= \left\langle 0^{2<w|1>}w1^{|w|}\,|1\right\rangle . \end{aligned}$$

This shows that $\mathcal{A}^* w \mathcal{A}^* \cap \mathcal{D} \neq \emptyset$ for all $w \in \mathcal{A}^*$. Therefore $\mathcal{D}$ is a dense fix-free code. Furthermore $\mathcal{D}$ is maximal prefix-free and maximal suffix-free. Let $w \in \mathcal{A}^+$. We show that w has a prefix in $\mathcal{D}$ or that there exists a word in $\mathcal{D}$ with prefix w.

Case 1: $<u|0>=<u|1>$ for some prefix u of w.
Let u be the prefix of w with $<u|0>=<u|1>$ and minimal length. Then u is in $\mathcal{D}$.

Case 2: $<u|0>\neq<u|1>$ for any prefix u of w.
We can suppose that $\langle w|0\rangle < \langle w|1\rangle$. Let $n :=<w|1>-<w|0>$. Then $w0^n$ is a word in $\mathcal{D}$ and has w as a prefix.

This shows that $\mathcal{D}$ is maximal prefix-free. The prove that $\mathcal{D}$ is maximal suffix-free follows the same steps.

Example 4 Let $\mathcal{A} = \{0,1\}$ and $C := \{10^n1 \,|\, n \geq 1\} \cup \{0\}$. Obviously $\mathcal{C}$ is a fix-free code. Furthermore we obtain:

$$\mathcal{A}^* 11 \mathcal{A}^* \cap \mathcal{C} = \emptyset\,.$$

This shows that $\mathcal{C}$ is an infinite thin fix-free code. We obtain for the Kraftsum of $\mathcal{C}$:

$$S(\mathcal{C}) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} - 1 = 1$$

For thin codes the following theorem holds:

Theorem 1 *Let $\mathcal{C} \subseteq \mathcal{A}^+$ be a thin set.*

(i) $\mathcal{C}$ is a maximal prefix-free code $\Leftrightarrow$ $\mathcal{C}$ is prefix-free and a maximal code.

(ii) $\mathcal{C}$ is a maximal suffix-free code $\Leftrightarrow$ $\mathcal{C}$ is suffix-free and a maximal code.

(iii) $\mathcal{C}$ is a maximal fix-free code.
$\Leftrightarrow$ $\mathcal{C}$ is fix-free and a maximal code.
$\Leftrightarrow$ $\mathcal{C}$ is a maximal prefix-free and a maximal suffix-free code.

A proof of the theorem can be found in [11]. Furthermore we will prove the theorem for finite prefix-free codes at the end of the next section. However, in general the theorem does not hold in for infinite codes as the next example shows.

Example 5 Let $\mathcal{A} := \{0,1\}$ and

$$X := \{u10^{|u|} \,|\, u \in \mathcal{A}^*\}\,.$$

It is easy to verify, that $\mathcal{X}$ is a suffix-free code, but not a prefix-free code. For example $1 \in \mathcal{X}$ is a prefix of $110 \in \mathcal{X}$. ¿From Proposition 9 follows that $\mathcal{Y} := \mathcal{X} - \mathcal{X}\mathcal{A}^+$ is a prefix-free code. Since $\mathcal{X}$ is not a prefix-free code it follows, that $\mathcal{Y} \neq \mathcal{X}$. Let $w \in \mathcal{A}^+$. Then we have $w10^{|w|} \in w\mathcal{A}^* \cap \mathcal{X}$. It follows that:

$$\emptyset \neq w\mathcal{A}^* \cap \mathcal{X} \subseteq \mathcal{A}^* w\mathcal{A}^* \cap \mathcal{X} \quad \text{for all} \;\; w \in \mathcal{A}^*\,.$$

Especially $\mathcal{X}$ is a dense code and $w\mathcal{A}^* \cap \mathcal{X}\mathcal{A}^* \neq \emptyset$ for all $w \in \mathcal{A}^*$. By Proposition 9 we have $\mathcal{X}\mathcal{A}^* = \mathcal{Y}\mathcal{A}^*$. Therefore we obtain:

$$w\mathcal{A}^* \cap \mathcal{Y}\mathcal{A}^* \neq \emptyset \quad \text{for all} \;\; w \in \mathcal{A}^*\,.$$

The equation above means, that for any word $w \in \mathcal{A}^*$ the word w is a prefix of a word in $\mathcal{Y}$ or that there exists a word in $\mathcal{Y}$ which is a prefix of w. Therefore the code $\mathcal{Y}$ is a maximal prefix-free code. Indeed $\mathcal{Y}$ is not a maximal code, because $\mathcal{X} \neq \mathcal{Y}$ and $\mathcal{Y} \subseteq \mathcal{X}$. Furthermore $\mathcal{Y}$ is a maximal fix-free code, since $\mathcal{X}$ is suffix-free. However, obviously $\mathcal{Y}$ is not a maximal suffix-free code.

The next lemma gives us the relationship between the Kraftsum of a prefix- suffix- or fix-free code, respectively and its shadow on the n-th level. For $x, y \in \mathcal{A}^*$ and $n \in \mathbb{N}_0$ we define the sets $I_n(x,y) \subseteq \mathcal{A}^n$ and $I(x,y) \subseteq \mathcal{A}^*$ as:

$$\begin{aligned} I_n(x,y) &:= \{z \in \mathcal{A}^n | x \text{ is prefix of } z \text{ and } y \text{ is suffix of } z\} = x\mathcal{A}^{n-|x|} \cap \mathcal{A}^{n-|y|}y\,, \\ I(x,y) &:= \bigcup_{n=0}^{\infty} I_n(x,y) = x\mathcal{A}^* \cap \mathcal{A}^* y\,. \end{aligned}$$

Lemma 8 *Let $|\mathcal{A}| = q \geq 2$, $x, y \in \mathcal{A}^*$ and $\mathcal{X} \subseteq \mathcal{A}^*$.*

(i) For any $N \in \mathbb{N}$ we have:

$$\begin{array}{lcl} \mathcal{X} \text{ is prefix-free} & \Leftrightarrow & |\Delta_P^n(\mathcal{X})| = q^n \cdot S_n(\mathcal{X}) \quad \forall\, n \geq N \\ \mathcal{X} \text{ is suffix-free} & \Leftrightarrow & |\Delta_S^n(\mathcal{X})| = q^n \cdot S_n(\mathcal{X}) \quad \forall\, n \geq N \\ \mathcal{X} \text{ is fix-free} & \Leftrightarrow & |\Delta_S^n(\mathcal{X})| = |\Delta_P^n(\mathcal{X})| = q^n \cdot S_n(\mathcal{X}) \quad \forall\, n \geq N \end{array}$$

(ii) If $\mathcal{X} \subseteq \bigcup_{l=0}^{N} \mathcal{A}^l$ for some $N \in \mathbb{N}_0$ and $n \geq N$ then:

$$\begin{array}{lcl} \mathcal{X} \text{ is prefix-free} & \Leftrightarrow & |\Delta_P^n(\mathcal{X})| = q^n \cdot S(\mathcal{X}) \\ \mathcal{X} \text{ is suffix-free} & \Leftrightarrow & |\Delta_S^n(\mathcal{X})| = q^n \cdot S(\mathcal{X}) \\ \mathcal{X} \text{ is fix-free} & \Leftrightarrow & |\Delta_S^n(\mathcal{X})| = |\Delta_P^n(\mathcal{X})| = q^n \cdot S(\mathcal{X}) \end{array}$$

(iii)
$$|\Delta_B^n(x)| = \begin{cases} 0 & \text{for} \quad |x| > n \\ 2 \cdot q^{n-|x|} - q^{n-2|x|} & \text{for} \quad 2|x| \leq n \\ 2 \cdot q^{n-|x|} - 1 & \text{for} \quad 2|x| > n \text{ and} \\ & x_{n-|x|+1} \dots x_{|x|} = x_1 \dots x_{2|x|-n} \\ 2 \cdot q^{n-|x|} & \text{for} \quad 2|x| > n \text{ and} \\ & x_{n-|x|+1} \dots x_{|x|} \neq x_1 \dots x_{2|x|-n} \end{cases}$$

(iv)
$$|I_n(x,y)| = \begin{cases} 0 & \text{for} \quad n < |x| \text{ or } n < |y| \\ 0 & \text{for} \quad n \geq |x|, |y|, n \leq |x| + |y| \text{ and} \\ & x_{n-|y|+1} \dots x_{|x|+|y|-n+1} \neq y_1 \dots y_{|x|+|y|-n} \\ 1 & \text{for} \quad n \geq |x|, |y|, n \leq |x| + |y| \text{ and} \\ & x_{n-|y|+1} \dots x_{|x|+|y|-n+1} = y_1 \dots y_{|x|+|y|-n} \\ q^{n-|x|-|y|} & \text{for} \quad n \geq |x| + |y| \end{cases}$$

(v) If $\mathcal{X}$ is fix-free:

$$\begin{aligned} |\Delta_B^n(\mathcal{X})| &= |\Delta_P^n(\mathcal{X})| + |\Delta_S^n(\mathcal{X})| - |\Delta_P^n(\mathcal{X}) \cap \Delta_S^n(\mathcal{X})| \\ &= 2 \cdot S_n(\mathcal{X}) - \sum_{x,y \in \mathcal{X}, |x|,|y| \leq n} |I_n(x,y)| \end{aligned}$$

Proof:

(i) Let $\mathcal{X} \subseteq \mathcal{A}^*$ be prefix-free and $n \geq N$. If $x, y \in \mathcal{X}$ with $x \neq y$ and $|x|, |y| \leq n$, then the sets $x\mathcal{A}^{n-|x|}$ and $y\mathcal{A}^{n-|y|}$ are disjoint. It follows that the sets $(\mathcal{X} \cap \mathcal{A}^l)\mathcal{A}^{n-l}$ and $(\mathcal{X} \cap \mathcal{A}^k)\mathcal{A}^{n-k}$ are also disjoint for $l < k \leq n$. Therefore we obtain:

$$
\begin{aligned}
|\Delta_P^n(\mathcal{X})| &= |\bigcup_{l=0}^{n}(\mathcal{X}\cap\mathcal{A}^l)\mathcal{A}^{n-l}| = \sum_{l=0}^{n}|\mathcal{X}\cap\mathcal{A}^l|\cdot|\mathcal{A}^{n-l}| \\
&= q^n\cdot\sum_{l=0}^{n}|\mathcal{X}\cap\mathcal{A}^l|q^{-l} = q^n\cdot S_n(\mathcal{X})\,.
\end{aligned}
$$

Let $\mathcal{X}\subseteq\mathcal{A}^*$ and $x,y\in\mathcal{X}$ with $x\neq y$ such that x is a prefix of y. Let $n:=\max\{N,|y|\}$, then $\emptyset\neq y\mathcal{A}^{n-|y|}\subseteq(\mathcal{X}\cap\mathcal{A}^{|x|})\mathcal{A}^{n-|x|}\cap(\mathcal{X}\cap\mathcal{A}^{|y|})\mathcal{A}^{n-|y|}$. Therefore it follows:

$$
|\Delta_P^n(\mathcal{X})| = |\bigcup_{l=0}^{n}(\mathcal{X}\cap\mathcal{A}^l)\mathcal{A}^{n-l}| < q^n\cdot\sum_{l=0}^{n}|\mathcal{X}\cap\mathcal{A}^l|q^{-l} = q^n\cdot S_n(\mathcal{X})\,.
$$

This shows (i) for prefix-free sets. The proof for suffix-free sets follows the same steps.

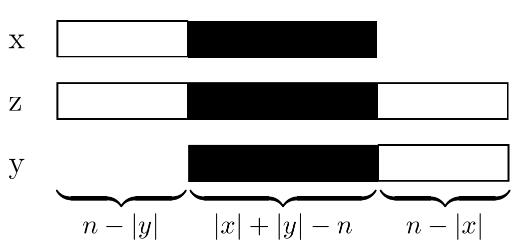
(ii) Let $\mathcal{X}\subseteq\bigcup_{l=0}^{N}\mathcal{A}^l$. Since $S_n(\mathcal{X})=S(\mathcal{X})$ for all $n\geq N$, part (ii) follows from part (i).

(iii) Since $\{x\}$ is a fix-free set for all $x\in\mathcal{A}^*$, part (iii) follows from (iv) and (v).

(iv) Let $|x|>n$ or $|y|>n$. Obviously there does not exist a word of length n, which has x as a prefix and y as a suffix. Therefore we obtain $|I_n(x,y)|=0$ for $\max\{|x|,|y|\}>n$.

If $|x|+|y|\geq n$ and $z\in\mathcal{A}^n$, such that x is a prefix of z and y is a suffix of z, the picture below shows that

$$
z = x_1\dots x_{|x|}y_{|x|+|y|-n+1}\dots y_{|y|} = x_1\dots x_{n-|y|}y_1\dots y_{|y|}\,.
$$



In this case follows $|I_n(x,y)|=1$.

If $|x|+|y|\leq n$, then we obtain $I_n(x,y)=x\mathcal{A}^{n-|x|-|y|}y$. In this case follows $|I_n(x,y)|=q^{n-|x|-|y|}$.

(v) Let $\mathcal{X}$ fix-free. While $\Delta_B^n(\mathcal{X}) = \Delta_P^n(\mathcal{X}) \cup \Delta_S^n(\mathcal{X})$, we obtain

$$|\Delta_B^n(\mathcal{X})| = |\Delta_P^n(\mathcal{X})| + |\Delta_S^n(\mathcal{X})| - |\Delta_P^n(\mathcal{X}) \cap \Delta_S^n(\mathcal{X})|$$

By (i) follows $S_n(\mathcal{X})q^n = |\Delta_P^n(\mathcal{X})| = |\Delta_S^n(\mathcal{X})|$. Furthermore we have:

$$\Delta_P^n(\mathcal{X}) \cap \Delta_S^n(\mathcal{X}) = \bigcup_{\substack{x,y \in \mathcal{X} \\ |x|,|y| \leq n}} I_n(x,y).$$

Since $\mathcal{X}$ is fix-free, no word in $\mathcal{A}^n$ has two different prefixes in $\mathcal{X}$ or two different suffixes in $\mathcal{X}$. It follows:

$$I_n(x,y) \cap I_n(x',y') = \emptyset \quad \forall\, x,y,x',y' \in \mathcal{X} \text{ with } (x,y) \neq (x',y').$$

Therefore we obtain

$$|\Delta_P^n(\mathcal{X}) \cap \Delta_S^n(\mathcal{X})| = \sum_{\substack{x,y \in \mathcal{X} \\ |x|,|y| \leq n}} I_n(x,y)\,. \quad \textbf{q.e.d}$$

Let $(\mathcal{X}_n)_n \in \mathbb{N}$ be a sequence of sets with $X_n \in \mathcal{A}^*$ for all $n \in \mathbb{N}$. We write $X_n \uparrow \mathcal{X}$ if $X_1 \subseteq X_2 \subseteq X_3 \subseteq \ldots$ and $X = \bigcup_{n=1}^{\infty} \mathcal{X}_n$. And we write $\mathcal{X}_n \downarrow \mathcal{X}$ if $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$ and $X = \bigcap_{n=1}^{\infty} \mathcal{X}_n$.

Proposition 9 *If $\mathcal{X}_n$ is prefix-free for all $n \in \mathbb{N}$ and $\mathcal{X}_n \uparrow \mathcal{X}$, then $\mathcal{X}$ is prefix-free, too.*

Proof: Let us suppose, that there exists $x, y \in \mathcal{X}$ such that x is a prefix of y. Then there exists an $n \in \mathbb{N}$, such that $x, y \in \mathcal{X}_n$. This is a contradiction, because $\mathcal{X}_n$ is prefix-free. **q.e.d**

Obviously the proposition holds also for suffix-free and fix-free sets.

We finish this section with two lemmas, which deals with the construction of fix-free codes and which we will use in Chapter 2.

Lemma 10 *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}^*$ be fix-free sets. Then the set $\mathcal{X}\mathcal{Y}$ is also fix-free.*

Furthermore the lemma above holds also for suffix-free and prefix-free sets.

Proof: Obviously the lemma holds for $\mathcal{X} = \{e\}$ or $\mathcal{Y} = \{e\}$. Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}^+$ be prefix-free codes. Let us assume that xy is a prefix of $x'y'$, where $x, x' \in \mathcal{X}$, $y, y' \in \mathcal{Y}$ and $xy \neq x'y'$. It follows that either x is a proper prefix of x' , x' is a proper prefix of x or $x = x'$. Since $\mathcal{X}$ is a prefix-free code, we obtain that $x = x'$, but then y is a prefix of y'. This is a contradiction because also $\mathcal{Y}$ is prefix-free. Therefore $\mathcal{X}\mathcal{Y}$ is prefix-free. The proof for suffix-free follows the same way. **q.e.d**

Lemma 11 *Let $\mathcal{X} \subseteq \bigcup_{l=0}^{n-1} \mathcal{A}^l$, $\mathcal{Y}, \mathcal{Z} \subseteq \mathcal{A}^n$ be such that $\mathcal{X} \cup \mathcal{Y}$ is fix-free. If $\mathcal{X}' \subseteq \mathcal{X}$ such that:*

(1) every word in $\mathcal{Z}$ has a prefix in $\mathcal{X}'$ or no prefix in $\mathcal{X}$,

(2) every word in $\mathcal{Z}$ has a suffix in $\mathcal{X}'$ or no suffix in $\mathcal{X}$,

then the set $(\mathcal{X} - \mathcal{X}') \cup \mathcal{Y} \cup \mathcal{Z}$ is fix-free.

Proof: By symmetry, it is sufficient to prove that (1) implies that $(\mathcal{X}-\mathcal{X}') \cup \mathcal{Y} \cup \mathcal{Z}$ is prefix-free. Obviously the lemma holds, if $\mathcal{X}' = \emptyset$ or $\mathcal{X} = \mathcal{X}'$. If $e \in \mathcal{X}$ then $\mathcal{X} = \{e\}$ and therefore $\mathcal{X}' = \emptyset$ or $\mathcal{X} = \mathcal{X}'$. Let $\emptyset \neq \mathcal{X}' \neq \mathcal{X}$ and suppose that $(\mathcal{X} - \mathcal{X}') \cup \mathcal{Y} \cup \mathcal{Z}$ is not prefix-free. Then there exists $z \in \mathcal{Z}$ and $x \in \mathcal{X} - \mathcal{X}'$ such that x is a prefix of z. By (1) follows, that z has a prefix x' in $\mathcal{X}'$. Since $x \neq x'$ and since both words x and x' are prefixes of z, it follows that either x is a proper prefix of x' or x' is a proper prefix of x. This is a contradiction, because $x, x' \in \mathcal{X}$ and $\mathcal{X}$ is prefix-free. **q.e.d**

1.3 The Kraftinequality for prefix-free codes

In this section we will show the Kraft-McMillan inequality for prefix-free codes (see McMillan [1]) and related results which can be found for example in [11].

Defenition 1
A Map $\pi : \mathcal{A}^* \longrightarrow \mathbb{R}^{\geq 0}$ *is called a Bernoulli Distribution on* $\mathcal{A}^*$ *if:*

(1) $\pi(xy) = \pi(x)\pi(y) \quad \forall\, x, y \in \mathcal{A}^*$

(2) $\pi(e) = 1$

(3) $\sum\limits_{x\in\mathcal{A}} \pi(a) = 1$

π *is called positive if* $\pi(a) \neq 0 \quad \forall\, a \in \mathcal{A}$

¿From (1) and (2) follows $\pi(x) = \prod\limits_{k=1}^{n} \pi(x_k)$ for $x = x_1 \ldots x_n$ Therefore π is unique determined by its values on $\mathcal{A}$. If π is positive, we obtain $\pi(x) \neq 0$ for all $x \in A^*$. It follows by (1) and (2), that a positive Bernoulli distribution is a monoidhomomorphism from $\mathcal{A}^*$ into $(\mathbb{R}^{>0}, 1, \cdot)$. For an arbitrary Bernoulli distribution π and $n \in \mathbb{N}_0$ we obtain:

$$\sum_{x\in\mathcal{A}^n} \pi(x) = \sum_{x\in\mathcal{A}^{n-1}} \pi(x) \underbrace{\sum_{x\in\mathcal{A}} \pi(a)}_{=1} = \sum_{x\in\mathcal{A}^{n-1}} \pi(x) = \ \ldots\ = 1. \tag{1.2}$$

Thus $\pi_{|_{A^n}}$ is a probability distribution on $\mathcal{A}^n$.

Let $\mathcal{M}$ be an arbitrary set. A *measure* on $\mathcal{M}$ is a map $\pi : \mathcal{P}(\mathcal{M}) \longrightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$ with the properties:

(1) $\pi(\emptyset) = 0$,

(2) If $\mathcal{X}_1, \mathcal{X}_2, \ldots \subseteq \mathcal{M}$ are pairwise disjoint, then $\pi\Big(\bigcup\limits_{n=1}^{\infty} \mathcal{X}_n\Big) = \sum\limits_{n=1}^{\infty} \pi(\mathcal{X}_n)$.
(σ-additivity)

Furthermore, $\mathcal{X} \subseteq Y \subseteq \mathcal{M}$ satisfy the inequality $\pi(\mathcal{X}) \leq \pi(\mathcal{Y})$ and if $\pi(\mathcal{Y}) < \infty$, then $\pi(\mathcal{Y} - \mathcal{X}) = \pi(\mathcal{Y}) - \pi(\mathcal{X})$.

Proposition 12 *Let $|\mathcal{A}| = q \geq 2$, $\mathcal{X} \subseteq \mathcal{A}^*$ and $\pi : \mathcal{A}^* \to \mathbb{R}^{\geq 0}$ be a Bernoulli distribution. If we define $\pi(X) := \sum_{x\in X} \pi(x)$, then $\pi : \mathcal{P}(\mathcal{A}^*) \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ is a measure on $\mathcal{P}(\mathcal{A}^*)$.*

Proof: Obviously $\pi(\emptyset) = 0$. Let $\mathcal{X}_1, \mathcal{X}_2, \ldots \subseteq \mathcal{M}$ pairwise disjoint, then:

$$\pi(\dot{\bigcup_{n\in\mathbb{N}}} \mathcal{X}_n) = \sum_{x\in \dot{\bigcup}_{n\in\mathbb{N}} \mathcal{X}_n} \pi(x) = \sum_{n=0}^{\infty} \sum_{x\in X_n} \pi(x) = \sum_{n=0}^{\infty} \pi(X_n)$$

This shows the σ-additivity of π. **q.e.d**

While π is a measure on $\mathcal{P}(\mathcal{A}^*)$, it has the following properties:

For each sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(\mathcal{A}^*)$ with $\mathcal{X}^n \uparrow \mathcal{X}$ the equation $\lim_{n\to\infty} \pi(\mathcal{X}_n) = \pi(\mathcal{X})$ holds. This means π is continues from below. (1.3)

For each sequence $(\mathcal{X}_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(\mathcal{A}^*)$ with $\mathcal{X}^n \downarrow \mathcal{X}$ and $\mathcal{X}_n < \infty$ for at least one $n \in \mathbb{N}$, the equation $\lim_{n\to\infty} \pi(\mathcal{X}_n) = \pi(\mathcal{X})$ holds. This means π is continues from above. (1.4)

The inequality $\pi(\bigcup_{n\in\mathbb{N}} \mathcal{X}_n) \leq \sum_{n=0}^{\infty} \pi(\mathcal{X}_n)$ holds for all sequences $(\mathcal{X}_n)_{n\in\mathbb{N}}$ in $\mathcal{P}(\mathcal{A}^*)$. (1.5)

A proof of the properties above can be found for example in [24].

The next example shows that the Kraftsum $S : \mathcal{P}(\mathcal{A}^*) \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ can be obtained from a positive Bernoulli distribution. Especially the map S is measure on $\mathcal{P}(\mathcal{A}^*)$.

Example 6 Let $|\mathcal{A}| = q \geq 2$. We define $\pi_K : \mathcal{A}^* \to \mathbb{R}^+$ as the (unique) positive Bernoulli distribution, given by the uniform distribution on $\mathcal{A}$. This means $\pi_K(a) := \frac{1}{q}$ for all $a \in \mathcal{A}$. It follows $\pi_K(x) = \pi_K(x_1) \dots \pi_K(x_{|x|}) = q^{-|x|}$ for all $x \in \mathcal{A}^+$ with $x = x_1 \dots x_{|x|}$ and $x_1, \dots, x_{|x|} \in \mathcal{A}$. Let $\mathcal{X} \subseteq \mathcal{A}^*$, then we obtain:

$$\pi_K(\mathcal{X}) = \sum_{x \in \mathcal{X}} q^{-|x|} = \sum_{l=0}^{\infty} \sum_{x \in \mathcal{X} \cap \mathcal{A}^l} q^{-l} = \sum_{l=0}^{\infty} |\mathcal{X} \cap \mathcal{A}^l| \cdot q^{-l} = S(\mathcal{X})$$

Lemma 13 *Let $\mathcal{C} \subseteq \mathcal{A}^+$ be a code and π be a Bernoulli distribution. Then $\pi(\mathcal{C}) \leq 1$. Especially $S(\mathcal{C}) \leq 1$ for every code $\mathcal{C} \subseteq \mathcal{A}^+$.*

Proof: Let $\mathcal{C} \subseteq \mathcal{A}^+$ be a code and π be a Bernoulli Distribution. We claim

$$\pi(\mathcal{C}^n) = \pi(\mathcal{C})^n \quad \text{for} \quad n \in \mathbb{N}\,. \tag{1.6}$$

Let $c_1, \dots, c_n, c'_1, \dots, c'_n \in \mathcal{C}$ such that $c_i \neq c'_i$ for some i. Since $\mathcal{C}$ is a code, we have $\{c_1 \cdot \dots \cdot c_n\} \cap \{c'_1 \cdot \dots \cdot c'_n\} = \emptyset$. With the σ-additivity of π, follows:

$$\begin{aligned}
\pi(\mathcal{C}^n) &= \pi\Big(\dot{\bigcup_{c_1,\dots,c_n \in \mathcal{C}}} \{c_1 \cdot \dots \cdot c_n\} \Big) \\
&\overset{\sigma\text{-Add.}}{=} \sum_{c_1,\dots,c_n \in \mathcal{C}} \pi(c_1) \cdot \dots \cdot \pi(c_n) \\
&= \underbrace{\sum_{c_1 \in \mathcal{C}} \pi(c_1)}_{=\pi(\mathcal{C})} \dots \underbrace{\sum_{c_1 \in \mathcal{C}} \pi(c_n)}_{=\pi(\mathcal{C})} = \pi(\mathcal{C})^n\,.
\end{aligned}$$

Next we claim:

$$\text{If } |\mathcal{C}| < \infty \text{ then } \pi(\mathcal{C}) \leq 1\,. \tag{1.7}$$

Let us suppose that $\pi(\mathcal{C}) = 1 + \epsilon$ for some $\epsilon > 0$. While $\mathcal{C}$ is finite, there exists $k \in \mathbb{N}$ with $\mathcal{C} \subseteq \mathcal{A} \mathbin{\dot\cup} \dots \mathbin{\dot\cup} A^k$. Then $\mathcal{C}^n \subseteq \mathcal{A} \mathbin{\dot\cup} \dots \mathbin{\dot\cup} A^{n \cdot k}$. It follows with the measure properties of π:

$$(1+\epsilon)^n = \pi(\mathcal{C})^n \underset{(1.6)}{=} \pi(\mathcal{C}^n) \underset{\substack{\pi \text{ is} \\ \text{measure}}}{\leq} \pi\Big(\bigcup_{l=1}^{n \cdot k} \mathcal{A}^l \Big) \underset{\sigma\text{-Add.}}{=} \sum_{l=1}^{n \cdot k} \underbrace{\pi(\mathcal{A}^l)}_{=1} = n \cdot k \text{ for all } n \in \mathbb{N}.$$

This is a contradiction, because for $\epsilon > 0, k \in \mathbb{N}$ there exists an $N \in \mathbb{N}$ with $(1+\epsilon)^n > n \cdot k \quad \forall n \geq N$. This proves (1.7).

We claim:

$$\pi(\mathcal{C}) \leq 1 \text{ is also true for } |\mathcal{C}| = \infty\,. \tag{1.8}$$

For $n \in \mathbb{N}$ let $\mathcal{C}_n := \mathcal{C} \cap \left(\bigcup_{k=0}^{n} \mathcal{A}^k \right)$, then $\mathcal{C}_n \uparrow \mathcal{C}$ and $|\mathcal{C}_n| < \infty$. From (1.7) follows $\pi(\mathcal{C}_n) \leq 1$ for all $n \in \mathbb{N}$ and since every measure is continues from bellow, we conclude:

$$\pi(\mathcal{C}) = \pi\left(\lim_{x \to \infty} C_n \right) = \lim_{x \to \infty} \pi(\mathcal{C}_n) \leq 1 \, .$$

This shows the lemma. Furthermore from Example 6 follows, that $S(\mathcal{C}) \leq 1$ for every code $\mathcal{C} \subseteq \mathcal{A}^+$. **q.e.d**

The next theorem shows, that for prefix-free codes and $\pi = S$, also the converse of Lemma 13 holds.

Theorem 2 (Kraft and McMillan [1]) *Let $|\mathcal{A}| = q \geq 2$ and $(\alpha_l)_{l \in \mathbb{N}}$ be a sequence of nonnegative integers. There exists a prefix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits to $(\alpha_l)_{l \in \mathbb{N}}$, if and only if $\sum_{l=1}^{\infty} \alpha_l q^{-l} \leq 1$.*

Furthermore the theorem holds also for suffix-free codes, in place of prefix-free codes.

Proof: If $\mathcal{C} \subseteq \mathcal{A}^+$ is a prefix-free code which fits $(\alpha_l)_{l \in \mathbb{N}}$, then from Lemma 13 follows, that $\sum_{l=1}^{\infty} \alpha_l q^{-l} = S(\mathcal{C}) \leq 1$. Let $(\alpha_l)_{l \in \mathbb{N}}$ be a sequence of nonnegative integers such that $0 < \sum_{l=0}^{\infty} \alpha_l q^{-l} \leq 1$. Since $1 \geq \alpha_1 \cdot q^{-1}$, it follows that $|\mathcal{A}| = q \geq \alpha_1$. Therefore we can choose a set $\mathcal{C}_1 \subseteq \mathcal{A}$ with $|\mathcal{C}_1| = \alpha_1$. Obviously this set is prefix-free.

Let $\mathcal{C}_n$ be a prefix-free set such that $\mathcal{C}_n \subseteq \bigcup_{l=1}^{n} \mathcal{A}^l$ and $\alpha_l = \left| \mathcal{C}_n \cap \mathcal{A}^l \right|$ for all $1 \leq l \leq n$. By Lemma 8 (ii) follows:

$$\left| \Delta_P^{n+1}(\mathcal{C}_n) \right| = q^{n+1} S(\mathcal{C}_n) = q^{n+1} \cdot \sum_{l=1}^{n} \alpha_l \cdot q^{-l}. \tag{1.9}$$

Since $\sum_{l=1}^{\infty} \alpha_l q^{-l} \leq 1$, we obtain:

$$\alpha_{n+1} \cdot q^{-(n+1)} \leq \sum_{l=n+1}^{\infty} \alpha_l q^{-l} \leq 1 - \sum_{l=1}^{n} \alpha_l \cdot q^{-l} = 1 - S(\mathcal{C}_n) \, . \tag{1.10}$$

By (1.9) follows:

$$\alpha_{n+1} \leq q^{n+1} - q^{n+1} S(\mathcal{C}_n) = \left| \mathcal{A}^{n+1} \right| - \left| \Delta_P^{n+1}(\mathcal{C}_n) \right| .$$

Thus we we can choose α_{n+1} codewords $c_1, \ldots, c_{\alpha_{n+1}} \in \mathcal{A}^{n+1}$, which are not in the prefix shadow of $\mathcal{C}_n$.[5] Then the set $\mathcal{C}_{n+1} := \mathcal{C}_n \cup \left\{c_1, \ldots, c_{\alpha_{n+1}}\right\}$ is prefix-free and fits to $\alpha_1, \ldots, \alpha_{n+1}$.

By induction we obtain prefix-free sets $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \ldots$, such that $\mathcal{C}_n$ fits to $(\alpha_1, \ldots, \alpha_n)$ for all $n \in \mathbb{N}$. Since $\mathcal{C}_n \uparrow \mathcal{C}$, it follows that $\mathcal{C} := \bigcup_{n=1}^{\infty} \mathcal{C}_n$ is a prefix-free code which fits to $(\alpha_l)_{l \in \mathbb{N}}$. This shows the theorem for prefix-free codes, the proof for suffix-free codes follows the same way. **q.e.d**

Let us now examine the relationship between maximal codes and Bernoulli distributions on $\mathcal{A}^*$. As a result of Lemma 13 it is easy to give a necessary condition for maximal codes.

Proposition 14 *Let $|\mathcal{A}| = q \geq 2$ and $\mathcal{C} \subseteq \mathcal{A}^*$ be a code. If there exists a positive Bernoulli distribution π on $\mathcal{A}^*$ with $\pi(\mathcal{C}) = 1$, then $\mathcal{C}$ is a maximal code.*

Proof: Let $\pi(\mathcal{C}) = 1$ for some positive Bernoulli distribution π on $\mathcal{A}^*$. Let us further assume, that $\mathcal{C}$ is not a maximal code. Then there exists a word $w \in \mathcal{A}^+ - \mathcal{C}$ such that $\mathcal{C} \cup \{w\}$ is a code. Since π is positive, we obtain that $\pi(w) > 0$. With Lemma 13 we obtain the contradiction:

$$1 \leq \pi\left(C \cup \{w\}\right) = \pi(\mathcal{C}) + \pi(w) = 1 + \pi(w) > 1\,.$$

Therefore $\mathcal{C}$ is a maximal code. **q.e.d**

For prefix-free codes the following converse of the proposition above holds.

Proposition 15 *Let $|\mathcal{A}| = q \geq 2$ and $\mathcal{C} \subseteq \mathcal{A}^*$ be a finite prefix-free code. $\mathcal{C}$ is a maximal prefix-free code if and only if $S(\mathcal{C}) = 1$. if and only if $\mathcal{C}$ is a maximal code.*

Furthermore the lemma above holds also for suffix-free codes.

Proof: Let $\mathcal{C} \subseteq \mathcal{A}^*$ be a finite prefix-free code with $S(\mathcal{C}) = 1$. Since the map S is a positive Bernoulli distribution on $\mathcal{A}^*$, it follows from Proposition 14, that $\mathcal{C}$ is maximal.

[5] If we order the finite alphabet in some arbitrary linear ordering, we can order the words in $\mathcal{A}^{n+1}$ by the lexicographical (well-)ordering , which is forced by the linear ordering of $\mathcal{A}$. Then we can choose the words $c_1, \ldots, c_{\alpha_{n+1}}$ in ascending order and avoid in this way some dubious choice principles. Especially the words which are chosen in the next step are all bigger than the words which were chosen before.

Let $\mathcal{C} \subseteq \bigcup_{l=1}^{n}$ be a finite prefix-free code with $S(\mathcal{C}) < 1$. ¿From Lemma 8 follows that $|\Delta_P^n(\mathcal{C})| = q^n S(\mathcal{C}) < q^n = |\mathcal{A}^n|$. We conclude that there exists a word $w \in \mathcal{A}^n$, which is not in the prefix-shadow of $\mathcal{C}$, i.e. w has not a prefix in $\mathcal{C}$. Obviously the set $\mathcal{C} \cup \{w\}$ is a prefix-free code. Therefore $\mathcal{C}$ is not a maximal prefix-free code. This shows that $S(\mathcal{C}) = 1$ if $\mathcal{C}$ is maximal prefix-free.

Obviously any code which is maximal and prefix-free is also a maximal prefix-free code. Thus we have shown:

$\mathcal{C}$ is maximal prefix-free $\Rightarrow S(\mathcal{C}) = 1 \Rightarrow \mathcal{C}$ is maximal $\Rightarrow \mathcal{C}$ is maximal prefix-free.
q.e.d

The proposition above shows Theorem 1 (i) for finite prefix-free codes. However the next theorem gives us a more general reversal of Proposition 14 for thin codes.

Theorem 3 *Let $\mathcal{C} \subseteq \mathcal{A}^*$ be a thin code. Then the following properties are all equivalent.*

(i) $\mathcal{C}$ is maximal.

(ii) $\pi(\mathcal{C}) = 1$ for every positive Bernoulli distribution π on $\mathcal{A}^*$.

(iii) There exists a positive Bernoulli distribution π on $\mathcal{A}^*$ with $\pi(\mathcal{C}) = 1$.

A proof of the theorem above can be found in [11]. For dense codes the theorem above is in general wrong. For example, it is shown in [11] that although the Dyckcode $\mathcal{D}$ in Example 3 has Kraftsum one, $\pi(\mathcal{D}) < 1$ for any other positive Bernoulli distribution π. On the other hand $\mathcal{D}$ is a maximal code, because of $S(\mathcal{D}) = 1$.

1.4 Kraftsums of fix-free codes and the $\frac{3}{4}$-conjecture

One might ask the question, wether Kraft's Theorem 2 holds for fix-free codes, as well. We will answer this question in general with no. However, the first part of Theorem 2 which is Lemma 13 for $\pi = S$, holds for all codes, i.e. for fix-free codes. If $\mathcal{C} := \mathcal{A}^n$ for some $n \in \mathbb{N}$ then $\mathcal{C}$ is a (maximal) fix-free code with $S(\mathcal{C}) = 1$. This shows, that 1 is the smallest number γ, such that $S(\mathcal{C}) \leq \gamma$ for all fix-free codes $\mathcal{C} \subseteq \mathcal{A}^*$. Furthermore Lemma 17 below in this section, shows that for any $0 < \gamma \leq 1$ there exists a fix-free code with Kraftsum equal to γ. Other construction of fix-free codes with Kraftsum 1 , which are especially maximal fix-free codes, can be found in [11] and [25]. The question rises, if there exists a number $0 < \gamma \leq 1$ for which the other direction of Theorem 2 holds for fix-free codes. More precisely: Does there exist a number γ such that $\sum\limits_{l=1}^{\infty} \alpha_l q^{-l} \leq \gamma$ imply the existence of a fix-free code which fits to $(\alpha_l)_{l\in\mathbb{N}}$ and which is the smallest possible γ? In [5] Ahlswede, Balkenhol and Khachatrian gave the conjecture below for binary codes and finite sequences, which was generalized by Harada and Kobayashi in [6] to arbitrary (finite) alphabets and infinite sequences in the form given below.

Conjecture 1 ($\frac{3}{4}$-conjecture) *Let $|\mathcal{A}| = q \geq 2$ and $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers, then $\sum\limits_{l=1}^{\infty} \alpha_l q^{-l} \leq \frac{3}{4}$ implies the existence of a fix-free $\mathcal{C} \subseteq \mathcal{A}^*$ which fits to $(\alpha_l)_{l\in\mathbb{N}}$.*

The next lemma shows that for every number bigger than $\frac{3}{4}$ the conjecture above can not hold. The lemma was first showed by Ahlswede, Balkenhol and Khachatrian in [5] for binary codes and finite sequences and it was generalized for arbitrary (finite) alphabets and infinite sequences by Harada and Kobayashi in [6].

Lemma 16 *Let $|\mathcal{A}| = q \geq 2$. For every $\varepsilon > 0$, there exists a finite sequence $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with $\sum\limits_{l=1}^{n} \alpha_l q^{-l} \leq \frac{3}{4} + \varepsilon$, such that for every fix-free code $\mathcal{X} \subseteq \bigcup\limits_{l=1}^{n} \mathcal{A}^l$, there exists some $1 \leq l \leq n$ with $|\mathcal{X} \cap \mathcal{A}^l| < \alpha_l$.*

Proof: Let $|\mathcal{A}| = q \geq 2$. It is sufficient to show the lemma for $0 < \varepsilon < \frac{1}{4}$.
Let $m \in \mathbb{N}$ with $\varepsilon \cdot q^m > 2$. We can choose $\alpha_m \in \mathbb{N}$ such that $q^m < 2\alpha_m < q^m + \epsilon \cdot q^m = q^m(1+\varepsilon)$ holds. It follows:

$$\frac{1}{2} < \alpha_m \cdot q^{-m} < \frac{1}{2} + \frac{\epsilon}{2}. \tag{1.11}$$

Let $n \in \mathbb{N}$, such that $n \geq 2m$ and $2\varepsilon q^n > 4$. Then we can choose a number $\alpha_n \in \mathbb{N}$ with

$$q^n < 4\alpha_n < q^n + 2\epsilon q^n = q^n(1 + 2\epsilon)\,.$$

We obtain for α_n:

$$\frac{1}{4} < \alpha_n \cdot q^{-n} < \frac{1}{4} + \frac{\varepsilon}{2}. \tag{1.12}$$

If we define $\alpha_l = 0$ for all $l \in \{1, \ldots, m-1, m+1, \ldots, 2m, \ldots, n-1\}$ we obtain for $\alpha_1, \ldots \alpha_n$ the desired property

$$\frac{3}{4} < \sum_{l=1}^{n} \alpha_l q^{-l} < \frac{3}{4} + \varepsilon. \tag{1.13}$$

Let $\mathcal{C} \subseteq \mathcal{A}^n$ be a set with $|\mathcal{C}| = \alpha_m$. We obtain:

$$|\Delta_B^n(\mathcal{C})| = |\Delta_P^n(\mathcal{C}) \cup \Delta_S^n(\mathcal{C})| = |\Delta_P^n(\mathcal{C})| + |\Delta_S^n(\mathcal{C})| - |\Delta_P^n(\mathcal{C}) \cap \Delta_S^n(\mathcal{C})|\,. \tag{1.14}$$

While $\mathcal{C}$ is a one-level set, $\mathcal{C}$ is fix-free. Therefore it follows:

$$|\Delta_P^n(\mathcal{C})| = |\Delta_S^n(\mathcal{C})| = \sum_{x \in \mathcal{C}} q^{n-m} = \alpha_m \cdot q^{n-m}\,.$$

Since $n \geq 2m$, from Lemma 8 follows:

$$\Delta_P^n(\mathcal{C}) \cap \Delta_S^n(\mathcal{C}) = \mathcal{C}\mathcal{A}^{n-2m}\mathcal{C}\,.$$

It follows:

$$|\Delta_P^n(\mathcal{C}) \cap \Delta_S^n(\mathcal{C})| = |\mathcal{C}|^2 \cdot \left|\mathcal{A}^{n-2m}\right| = \alpha_m^2 \cdot q^{n-2m}\,.$$

By (1.14) follows:

$$\begin{array}{rrcl} & |\Delta_B^n(\mathcal{C})| & = & 2\alpha_m q^{n-m} - \alpha_m^2 q^{n-2m} \\ \Rightarrow & \dfrac{|\Delta_B^n(\mathcal{C})| + \alpha_n}{q^n} & = & 2\dfrac{\alpha_m}{q^m} - \left(\dfrac{\alpha_m}{q^m}\right)^2 + \dfrac{\alpha_n}{q^n} \\ & & = & \dfrac{\alpha_n}{q^n} + 1 - \left(1 - \dfrac{\alpha_m}{q^m}\right)^2 \\ & & \underset{(1.11),\,(1.12)}{>} & \dfrac{1}{4} + 1 - \left(1 - \dfrac{1}{2}\right)^2 = 1 \\ \Rightarrow & |\Delta_B^n(\mathcal{C})| + \alpha_n & > & q^n = |\mathcal{A}^n|\,. \end{array}$$

It follows that $\mathcal{D} \cap |\Delta_B^n(\mathcal{C})| \neq \emptyset$ for every set $\mathcal{D} \subseteq \mathcal{A}^n$ with $|\mathcal{D}| = \alpha_n$. We conclude that for such $\mathcal{D}$'s the set $\mathcal{C} \cup \mathcal{D}$ is not a fix-free code. Since $\mathcal{C}$ was chosen

arbitrarily, this shows that there exist no fix-free code $\mathcal{X}$ with $\left|\mathcal{X} \cap \mathcal{A}^l\right| = \alpha_l$ for all $1 \leq l \leq n$. **q.e.d**

Let $\alpha_1, \dots, \alpha_n$ be as in the proof above, $\gamma := \frac{3}{4} + \varepsilon \in]\frac{3}{4}, 1]$ and $s := \frac{3}{4} + \epsilon - (\alpha_m q^{-m} + \alpha_n q^{-n}) \in [0; \epsilon[$. There exists a sequence $(\beta_k)_{k \in \mathbb{N}}$ such that $s = \sum_{k=1}^{\infty} \beta_k q^{-k}$ and $\beta_k \in \{0, \dots, (q-1)\}$ for all $k \in \mathbb{N}$. If we define $\tilde{\alpha}_l := \alpha_l + \beta_l$ for $1 \leq l \leq n$ and $\tilde{\alpha}_l := \beta_l$ for $l > n$, then we obtain that there exists no fix-free code which fits to $(\tilde{\alpha}_l)_{l \in \mathbb{N}}$. Furthermore, if it is possible to write γ as a finite Kraftsum, then also the sequence $(\tilde{\alpha}_l)_{l \in \mathbb{N}}$ is finite. This shows:

Corollary 1 *Let $|\mathcal{A}| = q \geq 2$ and $\frac{3}{4} < \gamma$. There exists a sequence $(\alpha_l)_{l \in \mathbb{N}}$ of nonnegative integers with $\gamma = \sum_{l=1}^{\infty} \alpha_l q^{-l}$ such that for every fix-free code $\mathcal{X} \subseteq \mathcal{A}^*$ there exists an $l \in \mathbb{N}$ with $|\mathcal{X} \cap \mathcal{A}^l| < \alpha_l$.*

However the next lemma shows that for any $0 < \gamma \leq 1$ there exists a fix-free code with Kraftsum γ.

Lemma 17 *Let $|\mathcal{A}| = q \geq 2$. For every $0 < \gamma \leq 1$ there exists a fix-free code $\mathcal{X} \subseteq \mathcal{A}^+$ with $S(\mathcal{X}) = \gamma$.*

Proof: If $\gamma = 1$ then for every $n \in \mathbb{N}$ the fix-free code $C := \mathcal{A}^n$ has Kraftsum 1. Furthermore it is sufficient to show the lemma for $\mathcal{A} = \{0, \dots, q-1\}$. Let $0 < \gamma < 1$. There exists a sequence $(\beta_l)_{l \in \mathbb{N}}$ with $\gamma = \sum_{l=1}^{\infty} \beta_l q^{-l}$, $\beta_l \in \{0, \dots, q-1\}$ for all $l \in \mathbb{N}$. Furthermore the sequence is unique if we assume that for any $n \in \mathbb{N}$ there exists an $l \geq n$ with $\beta_l \neq q-1$. Let $\mathcal{C}_1 := \{0, \dots, \beta_1 - 1\}$, if $\beta_1 \geq 1$ and $C_1 := \{0\}$ if $\beta_1 = 0$. Furthermore we define for $l \geq 2$:

$$\mathcal{D} := \mathcal{A} - \mathcal{C}_1 \text{ and } \mathcal{C}_l := \mathcal{D}\mathcal{C}_1^{l-2}\mathcal{D} \subseteq \mathcal{A}^n .$$

Obviously the set $\mathcal{C} := \bigcup_{l=1}^{\infty} \mathcal{C}_l$ is a fix-free code, where $\mathcal{C} \cap \mathcal{A}^l = \mathcal{C}_l$ for all $l \in \mathbb{N}$. For $l \geq 2$, the number of codewords on the l-th level of $\mathcal{C}$ is given by:

$$\left|\mathcal{C} \cap \mathcal{A}^l\right| = |\mathcal{C}_l| = |\mathcal{D}| \cdot \left|\mathcal{C}_1^{l-2}\right| \cdot |\mathcal{D}| = (q - \beta_1)^2 \beta_1^{(l-2)} \text{ for } \beta_1 \neq 0 ,$$

$$\left|\mathcal{C} \cap \mathcal{A}^l\right| = (q-1)^2 \text{ for } \beta_1 = 0 .$$

Case 1: $\beta_1 \neq 0$ and $\beta_2 \leq (q - \beta_1)^2$

An easy derivation shows that for $\beta_1 \neq 0$ the global minimum of

$f(q) := \beta_1(q-\beta_1)^2 - q + 1$ is given by $q_{\min} = \frac{1}{2\beta_1} + \beta_1$ and that there exists no other local minima. If $0 < \beta_1 < q$ and $q \in \mathbb{N}$, it follows that:

$$\begin{array}{rl} & 0 \le f(\beta_1) \le \beta_1(q-\beta_1)^2 - q + 1\,, \\ \Rightarrow & q - 1 \le \beta_1(q-\beta_1)^2\,, \\ \Rightarrow & \beta_l \le q - 1 \le \beta_1(q-\beta_1)^l = \left|\mathcal{C} \cap \mathcal{A}^l\right| \;\forall\, l \ge 3 \end{array}.$$

Therefore we obtain $|\mathcal{C} \cap \mathcal{A}^n| \ge \beta_l$ for all $l \ge 1$. Thus we can choose a subset $\mathcal{X}$ of $\mathcal{C}$ with $|\mathcal{X} \cap \mathcal{A}^n| = \beta_l$ for all $l \in \mathbb{N}$. This gives us a fix-free Code with Kraftsum r.

Case 2: $\beta_2 \ge 2$ and $\beta_2 > (q-\beta_1)^2$.
We define $\mathcal{X}_1 := C_1$ and as long as

$$0 < \sum_{l=2}^{n} \left(q^{(n-l)}\beta_k - q^{(n-l)}(q-\beta_1)^2\beta_1^{(l-2)}\right)$$

we define $\mathcal{X}_n := \mathcal{C}_n$. Since $\beta_1 \ge 2$ and $q - 1 \ge \beta_l\,\forall n \in \mathbb{N}$, there exists an $n \ge 2$ such that the sum above is smaller than or equal to zero. Let N be the smallest of such numbers. It follows:

$$\begin{array}{rcl} |\mathcal{C}_N| & = & (q-\beta_1)^2\beta_1^{(N-2)} \\ & \ge & \beta_N + q\sum\limits_{l=2}^{N-1}\left(q^{(N-1-l)}\beta_l) - q^{(N-1-l)}(q-\beta_1)^2\beta_1^{(l-2)}\right) > 0 \end{array}$$

Thus we can choose $\beta_N + q\sum\limits_{l=2}^{N-1}\left(q^{(N-1-l)}\beta_k) - q^{(N-1-l)}(q-\beta_1)^2\beta_1^{(l-2)}\right)$ words from $\mathcal{C}_N$ to obtain an $\mathcal{X}_N \subseteq \mathcal{C}_N$. With this definition of $\mathcal{X}_1, \ldots, \mathcal{X}_n$ we obtain:

$$\begin{array}{rcl} \sum\limits_{l=1}^{N} |\mathcal{X}_l|\, q^{-l} & = & \frac{\beta_1}{q} + \sum\limits_{l=2}^{N-1}(q-\beta_1)^2\beta_1^{(l-2)}q^{-l} + \sum\limits_{l=2}^{N-1}\left(q^{-l}\beta_l - q^{-l}(q-\beta_1)^2\beta_1^{(l-2)}\right) + \beta_N \\ & = & \sum\limits_{l=1}^{N}\beta_l q^{-l} \end{array}$$

In the same way as in Case 1, it follows, that $\beta_l \le q - 1 \le \beta_1^{l-2}(q-\beta_1)^2 = |\mathcal{C}_k|$ for $l \ge 3$. Therefore we can choose for every $l > N$ a set $\mathcal{X}_l \subseteq \mathcal{C}_l$ with $|\mathcal{X}_l| = \beta_l$. It follows that $\mathcal{X} := \bigcup\limits_{l=1}^{\infty}\mathcal{X}_l \subseteq \mathcal{C}$ is fix-free and $S(\mathcal{X}) = \sum\limits_{l=1}^{\infty}\beta_l q^{-l} = \gamma$.

Case 3: $\beta_1 \le 1$. We obtain $\beta_l \le q - 1 \le (q-1)^2 = |\mathcal{C}_n|\;\forall l \ge 2$. Therefore we can choose for every $l \in \mathbb{N}$ a set $\mathcal{X}_l \subseteq \mathcal{C}_l$ with $|\mathcal{X}_l| = \beta_l$. For $\mathcal{X} := \bigcup\limits_{l=1}^{\infty}\mathcal{X}_l$ we obtain that $\mathcal{X}$ is fix-free and and $S(\mathcal{X}) = \sum\limits_{l=1}^{\infty}\beta_l q^{-l} = \gamma$. **q.e.d**

Let $|\mathcal{A}| = q \geq 2$. For a number $\gamma \in \mathbb{R}$ we define the following properties:

$$\begin{aligned}&\textit{For any sequence } (\alpha_l)_{l\in\mathbb{N}} \textit{ of nonnegative integers with } \sum_{l=1}^{\infty} \alpha_l q^{-l} \leq \gamma\,,\\ &\textit{there exists a fix-free } \mathcal{C} \subseteq \mathcal{A}^* \textit{ which fits to } (\alpha_l)_{l\in\mathbb{N}}.\end{aligned} \tag{1.15}$$

$$\begin{aligned}&\textit{For any sequence } (\alpha_l)_{l\in\mathbb{N}} \textit{ of nonnegative integers with } \sum_{l=1}^{\infty} \alpha_l q^{-l} < \gamma\,,\\ &\textit{there exists a fix-free } \mathcal{C} \subseteq \mathcal{A}^* \textit{ which fits to } (\alpha_l)_{l\in\mathbb{N}}.\end{aligned} \tag{1.16}$$

$$\begin{aligned}&\textit{For any } n \in \mathbb{N} \textit{ and finite sequence } (\alpha_1, \ldots, \alpha_n) \textit{ of nonnegative}\\ &\textit{integers with } \sum_{l=1}^{n} \alpha_l q^{-l} \leq \gamma\,, \textit{ there exists a fix-free } \mathcal{C} \subseteq \mathcal{A}^* \textit{ which}\\ &\textit{fits to } (\alpha_1, \ldots, \alpha_n).\end{aligned} \tag{1.17}$$

$$\begin{aligned}&\textit{For any } n \in \mathbb{N} \textit{ and finite sequence } (\alpha_1, \ldots, \alpha_n) \textit{ of nonnegative}\\ &\textit{integers with } \sum_{l=1}^{n} \alpha_l q^{-l} < \gamma\,, \textit{ there exists a fix-free } \mathcal{C} \subseteq \mathcal{A}^* \textit{ which}\\ &\textit{fits to } (\alpha_1, \ldots, \alpha_n).\end{aligned} \tag{1.18}$$

$$\begin{aligned}&\textit{For any sequence } (\alpha_l)_{l\in\mathbb{N}} \textit{ of nonnegative integers with the properties}\\ &\textit{that for every } n \in \mathbb{N} \textit{ there exists an } l \geq n \textit{ with } \alpha_l \neq 0 \textit{ and}\\ &\sum_{l=1}^{\infty} \alpha_l q^{-l} < \gamma\,, \textit{ there exists a fix-free } \mathcal{C} \subseteq \mathcal{A}^* \textit{ which fits to } (\alpha_l)_{l\in\mathbb{N}}.\end{aligned} \tag{1.19}$$

The next proposition shows the relation between the different properties above:

Proposition 18

(i) For $\gamma \in \mathbb{R}$ we have:

$$(1.15) \Leftrightarrow (1.17) \Rightarrow (1.16) \Leftrightarrow (1.18) \Leftrightarrow (1.19)$$

(ii) If there exists an γ with one of the property above we obtain:

$$\sup_{\gamma \text{ has } (1.15)} \gamma = \sup_{\gamma \text{ has } (1.16)} \gamma = \sup_{\gamma \text{ has } (1.17)} \gamma = \sup_{\gamma \text{ has } (1.18)} \gamma = \sup_{\gamma \text{ has } (1.19)} \gamma$$

and the suprema above have the properties (1.16), (1.18) and (1.19).

Proof:
(1.16) $\Rightarrow$ (1.18): This holds obviously.

(1.18) $\Rightarrow$ (1.19): Let $\gamma \in (0,1]$ be a real number with property (1.18). Let $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers such that for every $n \in \mathbb{N}$ there exists an $l \geq n$ with $\alpha_l \neq 0$ and $\sum\limits_{l=1}^{\infty} \alpha_l q^{-l} \leq \gamma$. It follows:

$$\sum_{l=1}^{n} \alpha_l q^{-l} < \gamma \quad \forall\, n \in \mathbb{N}\,. \tag{1.20}$$

While γ has property (1.18), it follows that

$$\text{for all } n \in \mathbb{N} \text{ there exists a fix-free } \mathcal{D} \subseteq \mathcal{A}^* \text{ which fits to } (\alpha_1, \ldots \alpha_n). \tag{1.21}$$

¿From the property of the sequence $(\alpha_l)_{l\in\mathbb{N}}$ follows, that there exists $n_1 < n_2 < n_3 < \ldots$ such that $\alpha_{n_l} \neq 0$ for all $l \in \mathbb{N}$ and $\alpha_n = 0$ for all $n \notin \{n_l | l \in \mathbb{N}\}$. We define for $l \in \mathbb{N}$:

$$\begin{aligned} \mathcal{T}(l) &:= \{\mathcal{D} \subseteq \mathcal{A}^* | \mathcal{D} \text{ is fix-free and fits to } (\alpha_1, \ldots, \alpha_{n_l})\} \\ \mathcal{T} &:= \bigcup_{l=1}^{\infty} \mathcal{T}(l) \cup \{\emptyset\} \\ &= \{\mathcal{D} \subseteq \mathcal{A}^* | \mathcal{D} \text{ is fix-free and fits to } (\alpha_1, \ldots, \alpha_n) \text{ for some } n \in \mathbb{N}_0\} \end{aligned}$$

Obviously $(\mathcal{T}, \subseteq)$ is a tree, where the l-th level is given by $\mathcal{T}(l)$, i.e. every node in $\mathcal{T}$ is a finite node. $\mathcal{T}$ is an ω-tree, because by (1.21) follows that $\mathcal{T}(l) \neq \emptyset$ for all $l \in \mathbb{N}$. Furthermore $|\mathcal{T}(l)| < \infty$ for all $l \in \mathbb{N}$, because every $\mathcal{D} \in \mathcal{T}(l)$ is a subset of the finite set $\bigcup\limits_{i=0}^{n_l} \mathcal{A}^i$. From Königs's Lemma 4 follows, that there exists an infinite branch in $\mathcal{T}$. This means, there exists $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subset \ldots$ with $\mathcal{D}_l \in \mathcal{T}(l)$

for all $l \in \mathbb{N}$. Especially $\mathcal{D}_l$ is fix-free and fits to $(\alpha_1, \ldots, \alpha_{n_l})$ for all $l \in \mathbb{N}$. Let $\mathcal{C} := \bigcup\limits_{l=1}^{\infty} \mathcal{D}_l$, then $\mathcal{C}$ is fix-free, because $\mathcal{D}_l \uparrow \mathcal{C}$ and obviously $\mathcal{C}$ fits to $(\alpha_l)_{l\in\mathbb{N}}$.

(1.19) ⇒ (1.16): Let $0 < \gamma \leq 1$ be a real number which fulfill (1.19). Then also (1.18) holds for γ. Let $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers with $\sum\limits_{l=1}^{\infty} \alpha_l q^{-l} < \gamma$. Either the sequence has the property in (1.19) or there exists an $n \in \mathbb{N}$ such that $\alpha_l = 0$ for all $l \geq n$. In the first case the existence of a fix-free code which fits to the sequence follows from (1.19) and in the second case the existence of the fix-free code follows from (1.18).

(1.15) ⇒ (1.17) ⇒ (1.18): This holds obviously.

(1.17) ⇒ (1.15): Let $0 < \gamma \leq 1$ such that (1.17) holds, then ,as shown above, also (1.19) holds for γ. Let $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers with $\sum\limits_{l=1}^{\infty} \alpha_l q^{-l} \leq \gamma$. If there exists an $n \in \mathbb{N}$ such that $\alpha_l = 0$ for all $l \geq$, there exists a fix-free code which fits the sequence by (1.17). Otherwise for every $n \in \mathbb{N}$ there exists an $l \geq n$ with $\alpha_l \neq 0$ and from (1.19) follows that there exists a fix-free code which fits to $(\alpha_l)_{l\in\mathbb{N}}$.

This shows part (i). Part (ii) follows from part (i). **q.e.d**

The next lemma shows that there exist a γ which fulfill (1.15) holds. The lemma was first proven by Ahlswede, Balkenhol and Khachatrian in [5] for binary codes and finite sequences. Harada and Kobayashi gave in [6] a proof of the lemma for arbitrary (finite) alphabets and infinite sequences.

Lemma 19 *Let $|\mathcal{A}| = q \geq 2$ and $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers. If $\sum\limits_{l=1}^{\infty} \alpha_l q^{-l} \leq \frac{1}{2}$, then there exists a fix-free $\mathcal{C} \subseteq \mathcal{A}^*$ which fits to $(\alpha_l)_{l\in\mathbb{N}}$.*

Proof: (The proof is very similar to the proof of theorem 2)

By the condition we obtain $\alpha_1 \frac{1}{q} \leq \frac{1}{2}$. Thus $\alpha_1 \leq \frac{q}{2} < |\mathcal{A}|$ and we can choose a set $\mathcal{C}_1 \subseteq \mathcal{A}$ with $|\mathcal{C}_1| = \alpha_1$. Obviously $\mathcal{C}_1$ is fix-free.

Let $\mathcal{C}_n \subseteq \bigcup_{k=0}^{n} \mathcal{A}^k$ be a fix-free set with $\left|\mathcal{C}_n \cap \mathcal{A}^l\right| = \alpha_l$ for all $1 \leq l \leq n$. By Lemma 8 we have:

$$\left|\Delta_P^{n+1}(\mathcal{C}_n)\right| = \left|\Delta_S^{n+1}(\mathcal{C}_n)\right| = \sum_{l=1}^{n} \alpha_l q^{n+1-l} \leq \frac{1}{2} \tag{1.22}$$

While $\sum_{l=1}^{n+1} \alpha_l q^{-l} \leq \sum_{l=1}^{\infty} \alpha_l q^{-l} \leq \frac{1}{2}$, it follows:

$$\left|\mathcal{A}^{n+1}\right| = q^{n+1} \geq 2\alpha_{n+1} + 2\sum_{l=1}^{n} \alpha_l q^{n+1-l}.$$

Thus we obtain:

$$\begin{aligned}
\left|\mathcal{A}^{n+1}\right| - 2\alpha_{n+1} \quad &\geq \quad 2\sum_{l=1}^{n} \alpha_l q^{n+1-l} \\
&\underset{(2.2)}{\geq} \quad 2\left|\Delta_P^{n+1}(\mathcal{C}_n)\right| \\
&\underset{(2.2)}{\geq} \quad \left|\Delta_P^{n+1}(\mathcal{C}_n)\right| + \left|\Delta_S^{n+1}(\mathcal{C}_n)\right| - \left|\Delta_P^{n+1}(\mathcal{C}_n) \cap \Delta_S^{n+1}(\mathcal{C}_n)\right| \\
&= \quad \left|\Delta_B^{n+1}(\mathcal{C}_n)\right|.
\end{aligned}$$

It follows that $|\mathcal{A}^{n+1}| \geq \left|\Delta_B^{n+1}(\mathcal{C}_n)\right| + \alpha_{n+1}$.

Therefore we can choose α_{n+1} words $c_1, \ldots, c_{\alpha_{n+1}} \in \mathcal{A}^{n+1}$, which are not in the $(n+1)$-th level bifix-shadow of $\mathcal{C}_n$. Thus $\mathcal{C}_{n+1} := C_n \cup \left\{c_1, \ldots, c_{\alpha_{n+1}}\right\}$ is fix-free and fits $(\alpha_1, \ldots, \alpha_{n+1})$.

Let $\mathcal{C} := \bigcup_{l=1}^{\infty} \mathcal{C}_l$, then $\mathcal{C}$ fits to $(\alpha_l)_{l\in\mathbb{N}}$. and since $\mathcal{C}_l \uparrow \mathcal{C}$, the code $\mathcal{C}$ is fix-free. **q.e.d**

¿From Lemma 16 and Lemma 19 follows, that there exists a $\gamma \in [\frac{1}{2}, \frac{3}{4}]$ which fulfill (1.15) and that for every $\gamma > \frac{3}{4}$ property (1.15) does not hold. This gives us the conjecture:

Conjecture 2

$$\sup_{\gamma \textit{ fullfil (1.15)}} \gamma = \frac{3}{4}$$

However, the conjecture above is weaker than the Conjecture 1, since from Proposition 18 follows that Conjecture 2 is equivalent to:

> For any sequence $(\alpha_l)_{l\in\mathbb{N}}$ of nonnegative integers with $\sum_{l=1}^{\infty} \alpha_l q^{-l} < \frac{3}{4}$, there exists a fix-free code which fits to $(\alpha_l)_{l\in\mathbb{N}}$. (1.23)

1.5 Extensions of fix-free codes

Let P be a property defined for sequences of nonnegative integers. We call P an *extension property* for sequences if

(1) for any finite sequence $(\alpha_1, \ldots, \alpha_n)$ which fulfill P also $(\alpha_1, \ldots, \alpha_{n-1})$ fulfill P,

(2) for any infinite sequence $(\alpha_l)_{l \in \mathbb{N}}$ for which P holds, also $(\alpha_1, \ldots, \alpha_n)$ fulfill P for all $n \in \mathbb{N}$.

We call P an σ*-extension* property, if further on, $(\alpha_1, \ldots, \alpha_n)$ has property P for all $n \in \mathbb{N}$ for a sequence $(\alpha_l)_{l \in \mathbb{N}}$ imply that also $(\alpha_l)_{l \in \mathbb{N}}$ fulfill P.

Let $\mathcal{M} \subseteq \mathcal{P}(\mathcal{A}^*)$. We call $\mathcal{M}$ an *extension class*, if the following properties hold for $\mathcal{M}$:

(1) $\emptyset \in \mathcal{M}$,

(2) if there exists $\mathcal{C} \in \mathcal{M}$ which fits to a finite sequence $(\alpha_1, \ldots, \alpha_n)$ then there exists a set $\mathcal{D} \in \mathcal{M}$ which fits to $(\alpha_1, \ldots, \alpha_{n-1})$,

(3) if there exists a set $\mathcal{C} \in \mathcal{M}$ which fits to a sequence $(\alpha_l)_{l \in \mathbb{N}}$ then for every $n \in \mathbb{N}$ there exists a set $\mathcal{C}_n \in \mathcal{M}$ which fits to the finite sequence $(\alpha_1, \ldots, \alpha_n)$.

Furthermore we call $\mathcal{M}$ an σ*-extension class*, if $\bigcup\limits_{l \in \mathbb{N}} \mathcal{C}_n \in \mathcal{M}$ for every ascending set sequence $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \ldots$ with $C_1, \mathcal{C}_2, \ldots \in \mathcal{M}$.

For example the classes of prefix-, suffix- and fix-free sets are all σ-extension classes. Let $0 < \gamma \leq 1$. For sequences of nonnegative integers. We define the properties P_γ and $P_{<\gamma}$ as follows:

$$P_\gamma : \quad \sum_{l=1}^{\infty} \alpha_l q^{-l} \leq \gamma \quad \text{and} \quad P_{<\gamma} : \quad \sum_{l=1}^{\infty} \alpha_l q^{-l} < \gamma \, .$$

Obviously P_γ and $P_{<\gamma}$ are σ-extension properties.

We denote with $\mathcal{I}(P), \mathcal{F}(P), \mathcal{M}(P)$ and $\mathcal{M}_F(P)$ the following sets:

> $\mathcal{I}(P)$ is defined as the set of all sequences which have property P.
>
> $\mathcal{F}(P)$ is defined as the set of all finite sequences which fulfill P.
>
> $\mathcal{M}(P)$ is defined as the class of all sets in $\mathcal{M}$ which fits to a sequence in $I(P)$.
>
> $\mathcal{M}_f(P)$ is defined as the class of all sets in $\mathcal{M}$ which fits to a sequence in $\mathcal{F}(P)$.

Obviously $\mathcal{M}_f(P) \subseteq \mathcal{M}(P)$. If P is an extension property and $\mathcal{M}$ an extension class, it is easy to verify that $\mathcal{M}_f(P)$ and $\mathcal{M}(P)$ are extension classes. Furthermore if $\mathcal{M}$ is an σ-extension class and P a σ-extension property, then $\mathcal{M}(P)$ is an σ-extension class, as well.

Let P an extension property. We call an extension class $\mathcal{M}$ a *P-simple extension class* if for $\mathcal{M}$ the following property holds:

Simple extension property: *Let $(\alpha_1, \ldots, \alpha_n) \in \mathcal{F}(P)$. If there exists a set in $\mathcal{M}$ which fits to $(\alpha_1, \ldots, \alpha_n)$, then for every $\mathcal{C} \in \mathcal{M}$ which fits to $(\alpha_1, \ldots, \alpha_{n-1})$ there exists an extension in $\mathcal{M}$ which fits to $(\alpha_1, \ldots, \alpha_n)$, i.e. the extension and $\mathcal{C}$ are a sets in $\mathcal{M}_f(P)$.* (1.24)

If $\mathcal{M}$ fulfill (1.24) for all sequences of nonnegative integers, then we call $\mathcal{M}$ a simple extension class. Property (1.24) means, that for a finite set $C \in \mathcal{M}$, the existence of an extension in $\mathcal{M}$ which fits to a sequence in $\mathcal{F}(P)$, does not depend on the words contained in $\mathcal{C}$, but on the values of $|\mathcal{C} \cap \mathcal{A}^l|$ for $l \in \mathbb{N}$. Therefore the following simple strategy is possible, for finding a set in a P-simple extension class $\mathcal{M}$ which fits to a sequence $(\alpha_1, \ldots, \alpha_n) \in \mathcal{F}(P)$:

1. Choose an arbitrary set $\mathcal{C}_1 \subseteq \mathcal{A}$ with $\mathcal{C}_1 \in \mathcal{M}$ and $|\mathcal{C}_1| = \alpha_1$.
2. If a set $\mathcal{C}_l \in \mathcal{M}$ which fits to $(\alpha_1, \ldots, \alpha_l)$ is already constructed, then choose $\mathcal{C}_{l+1}$ as an arbitrary extension of $\mathcal{C}_l$ in $\mathcal{M}$ which fits to $(\alpha_1, \ldots, \alpha_l)$.

If there exists at least one set in $\mathcal{M}$ which fits to $(\alpha_1, \ldots, \alpha_n)$, then from the simple extension property follows, that the construction above gives us after n

steps a set $\mathcal{C}_n \in \mathcal{M}$ which fits to $(\alpha_1, \dots, \alpha_n)$. Furthermore, if $\mathcal{M}$ is a P-simple σ-extension class for some σ-extension property P, then there exists a set in $\mathcal{M}$ which fits to a sequence $(\alpha_l)_{l\in\mathbb{N}}$ in $\mathcal{I}(P)$ if and only if the construction above doesn't stop. In this case the set $\mathcal{C} := \bigcup_{l=1}^{\infty} \mathcal{C}_l$ is a set in $\mathcal{M}$ which fits to $(\alpha_l)_{l\in\mathbb{N}}$.

Since the cardinality of the prefix-shadow on the $(n+1)$-th level of a prefix-free set $\mathcal{C} \subseteq \bigcup_{l=0}^{n} \mathcal{A}^l$ only depends on the Kraftsum of $\mathcal{C}$, it follows that the class of prefix-free sets is a simple σ-extension class. However, for a fix-free set $\mathcal{C} \subseteq \bigcup_{l=0}^{n} \mathcal{A}^l$ the bifix-shadow on the $(n+1)$-th level is given by:

$$|\Delta_B^{n+1}(\mathcal{C})| = 2|\Delta_P^{n+1}(\mathcal{C})| - \sum_{x,y\in\mathcal{C}} I_{n+1}(x,y) = 2|\Delta_P^{n+1}(\mathcal{C})| - |\Delta_P^{n+1}(\mathcal{C}) \cap \Delta_S^{n+1}(\mathcal{C})|.$$

In general the sum $\sum_{x,y\in\mathcal{C}} I_{n+1}(x,y)$ depends on the codewords contained in $\mathcal{C}$ and does not depends on the codeword lengths only.

The next example shows that, the class of fix-free sets is not a simple extension class.

Example 7 Let $\mathcal{A} = \{0,1\}$ and $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 2, \alpha_4 = 4, \alpha_l = 0$ for $l \geq 5$. For the Kratsum we obtain:

$$\sum_{l=1}^{\infty} \alpha_l \cdot \frac{1}{2}^l = \frac{1}{4} + \frac{2}{8} + \frac{4}{16} = \frac{3}{4}$$

$\mathcal{D} := \{00, 101, 110\}$ is a fix-free code which fits to $(\alpha_1, \alpha_2, \alpha_3)$. Since
$|\mathcal{A}_4 - \Delta_B^4(\mathcal{D})| = |\{1111, 0111, 1001\}| = 3 < 4 = \alpha_4$, it follows that there does not exist a fix-free $\mathcal{C} \supseteq \mathcal{D}$ which fits to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

Indeed $\mathcal{D}' := \{10, 000, 111, 1100, 0011, 0101, 1101\}$ is a fix-free code which fits to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

Moreover the example above shows, that the class of fix-free sets, is also not a $P_{\frac{3}{4}}$-simple extension class. The question arise if there exists extension properties P such the class of fix-free codes is a P-extension class? The proof of Lemma 19 shows for example that the the class of fix-free sets is a $P_{\frac{1}{2}}$-simple extension class. While for fix-free codes with Kraftsum smaller than or equal to $\frac{1}{2}$, there are for an extension, such less codewords necessary, that it is possible to ignore the value of $|\Delta_P^{n+1}(\mathcal{C}) \cap \Delta_S^{n+1}(\mathcal{C})|$.

Another example is the following property $P^*_{\frac{3}{4}}$ for sequences $(\alpha_l)_{l\in\mathbb{N}}$ of nonnegative integers:

$$P^*_{\frac{3}{4}} \quad : \quad \sum_{l=1}^{\infty} \alpha_l q^{-l} \leq \frac{3}{4} \quad \text{and} \quad \alpha_l \neq 0 \Rightarrow \alpha_{l+1} = \alpha_{l+2} = \ldots = \alpha_{2l-1} = 0$$

Obviously $P^*_{\frac{3}{4}}$ is an σ-extension property. Furthermore in the proof of Theorem 4 at the beginning of the next chapter, we will show that the class of fix-free sets is a $P^*_{\frac{3}{4}}$-simple extension class. We might ask for the supremum of numbers γ for which the class of fix-free sets is a P_γ-simple extension class. For the supremum γ^* follows that the class of fix-free sets is a $P_{<\gamma^*}$-simple extension class.

The next example shows that the class of fix-free codes is not a $P_{<\frac{3}{4}}$-simple extension class. Therefore it follows that $\gamma^* < \frac{3}{4}$.

Example 8 Let $\mathcal{A} := \{0, 1\}$ and $\alpha_1 = \alpha_2 = 0, \alpha_3 = 4, \alpha_4 = 1, \alpha_5 = 5, \alpha_l = 0$ for $l \geq 6$. We obtain for the Kraftsum of the sequence:

$$\sum_{l=1}^{\infty} \alpha_l \cdot \frac{1}{2}^l = \frac{4}{8} + \frac{1}{16} + \frac{5}{32} = \frac{23}{32} < \frac{3}{4}\,.$$

The set $\mathcal{D} := \{000, 111, 011, 001, 0101\}$ is a fix-free code and fits to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

Level 5	P	S	Level 5	P	S	Level 5	P	S
00000	x	x	01011	x	x	**10110**		
00001	x	x	01100	x		10111		x
00010	x		01101	x		11000		x
00011	x	x	01110	x		11001		x
00100	x		01111	x		**11010**		
00101	x	x	10000		x	11011		x
00110	x		10001		x	11100	x	
00111	x	x	**10010**			11101	x	
01000		x	10011		x	11110	x	
01001		x	**10100**			11111	x	x
01010	x		10101		x			

The tabular above shows that:

$$|\mathcal{A}^5 - \Delta^5_B(\mathcal{D})| = |\{10010, 10100, 10110, 11010\}| = 4 < 5 = \alpha_5\,.$$

It follows that there does not exist a fix-free code $\mathcal{D}$ with $\mathcal{C} \supseteq \mathcal{D}$ which fits to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$. However there exist fix-free codes which fits to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$. As an example:

$$\{000, 001, 010, 011, 1110, 10100, 10101, 10110, 10111, 11111\}$$

Since the class of fix-sets is a $P_{\frac{1}{2}}$-simple extension class, we give the following conjecture:

Conjecture 3

$$\frac{1}{2} = \max\{\gamma \mid \textit{the class of fix-free codes is a } P_\gamma\textit{-simple extension class}\}$$

Instead of searching for properties P of sequences for which the class of fix-free sets is a P-simple extension class, one might try to find a subclass $\mathcal{M}$ of fix-free sets such that $\mathcal{M}$ is a simple extension or a $P_{\frac{3}{4}}$-simple extension class. However in this survey we don't pay attention to this problem.

Let $\alpha_1, \ldots, \alpha_{n+1} \in \mathbb{N}_0$ and $\mathcal{D} \subseteq \mathcal{A}^*$ be a fix-free set which fits to $(\alpha_1, \ldots, \alpha_n)$. By Lemma 8 follows:

$$|\Delta_B^{n+1}(\mathcal{D})| = 2q^{n+1}S(\mathcal{D}) - |\Delta_P^{n+1}(\mathcal{D}) \cap \Delta_S^{n+1}(\mathcal{D})|\,.$$

Therefore the existence of an extension of $\mathcal{D}$ which fits to $(\alpha_1, \ldots, \alpha_{n+1})$ depends on the value of $|\Delta_P^{n+1}(\mathcal{D}) \cap \Delta_S^{n+1}(\mathcal{D})|$ and $(\alpha_1, \ldots, \alpha_{n+1})$. The next lemma shows, for which values of $|\Delta_P^{n+1}(\mathcal{D}) \cap \Delta_S^{n+1}(\mathcal{D})|$ an extension is possible, if the Kraftsum of the sequence is smaller than or equal to $\frac{3}{4}$. The following lemma can be found in [6].

Lemma 20 *Let $|\mathcal{A}| = q \geq 2$, $n > k$, $\alpha_1, \ldots, \alpha_k \in \mathbb{N}_0$, $\alpha_{k+1} = \ldots = \alpha_{n-1} = 0$ and $\alpha_n \in \mathbb{N}$ such that $\sum\limits_{l=0}^{n} \alpha_l \cdot q^{-l} \leq \frac{3}{4}$. Let $\mathcal{D}$ be a fix-free set which fits to $(\alpha_1, \ldots, \alpha_k)$.*

(i): If $\frac{|\Delta_P^n(\mathcal{D}) \cap \Delta_S^n(\mathcal{D})|}{q^n} \geq \begin{cases} \frac{\Delta_P^n(\mathcal{D})}{q^n} + \frac{\lfloor \frac{3}{4} q^n \rfloor}{q^n} - 1 & \textit{if } q \textit{ is odd} \\ \frac{\Delta_P^n(\mathcal{D})}{q^n} - \frac{1}{4} & \textit{if } q \textit{ is even} \end{cases}$,

then there exists a fix-free extension $\mathcal{C} \supseteq \mathcal{D}$ which fits to $(\alpha_1, \ldots, \alpha_{n+k})$.

(ii): If $\frac{|\Delta_P^n(\mathcal{D}) \cap \Delta_S^n(\mathcal{D})|}{q^n} \geq \left(\frac{\Delta_P^n(\mathcal{D})}{q^n}\right)^2$ *, then there exists a fix-free extension $\mathcal{C} \supseteq \mathcal{D}$ which fits to $(\alpha_1, \ldots, \alpha_{n+k})$.*

Proof:

(i): Let $(\alpha_1, \ldots, \alpha_{n+k}), \mathcal{D}$ be as in the Lemma. For even q and $n \geq 2$ we have $\frac{\lfloor \frac{3}{4} q^n \rfloor}{q^n} - 1 = -\frac{1}{4}$. Therefore it is sufficient to show that

$$\frac{|\Delta_P^n(\mathcal{D}) \cap \Delta_S^n(\mathcal{D})|}{q^n} \geq \frac{\Delta_P^n(\mathcal{D})}{q^n} + \frac{\lfloor \frac{3}{4} q^n \rfloor}{q^n} - 1 \tag{1.25}$$

imply the existence of a fix-free Code $\mathcal{C} \supseteq \mathcal{D}$ which fits to $(\alpha_1, \ldots, \alpha_n)$. We obtain from the conditions of $(\alpha_1, \ldots, \alpha_n)$:

$$\begin{aligned}
& \tfrac{3}{4} \geq \sum_{l=1}^{n} \alpha_l q^{-l} = \alpha_n q^{-n} + \sum_{l=1}^{k} \alpha_l q^{-l}\,, \\
\Rightarrow \quad & \tfrac{3}{4} q^n \geq \alpha_n + \sum_{l=1}^{k} \alpha_l q^{n-l} = \alpha_n + |\Delta_P^n(\mathcal{D})| \in \mathbb{N}\,, \\
\Rightarrow \quad & \lfloor \tfrac{3}{4} q^n \rfloor \geq \alpha_n + |\Delta_P^n(\mathcal{D})|\,.
\end{aligned}$$

By (1.25) it follows:

$$|\Delta_P^n(\mathcal{D}) \cap \Delta_S^n(\mathcal{D})| \geq 2|\Delta_P^n(\mathcal{D})| + \alpha_n - q^n\,.$$

While $|\Delta_B^n(\mathcal{D})| = 2|\Delta_P^n(\mathcal{D})| - |\Delta_P^n(\mathcal{D}) \cap \Delta_S^n(\mathcal{D})|$ (by Lemma 8) and $q^n = |\mathcal{A}^n|$, we conclude:

$$|\mathcal{A}^n| - |\Delta_B^n(\mathcal{D})| \geq \alpha_n\,.$$

Thus we can choose α_n different words $c_1, \ldots, c_{\alpha_n} \in \mathcal{A}^n$ of length n, which are not in the bifix-shadow of $\mathcal{D}$. Then $\mathcal{C} := \mathcal{D} \cup \{c_1, \ldots, c_{\alpha_n}\}$ is a fix-free Code with the desiered properties.

(ii): The function $f(x) := x^2$ is convex. Therefore we have:

$$x^2 \geq f'(\frac{1}{2})(x - \frac{1}{2}) + f(\frac{1}{2}) = x - \frac{1}{4} \geq x + \frac{\lfloor \frac{3}{4} q^n \rfloor}{q^n} - 1 \tag{1.26}$$

If $\frac{|\Delta_P^n(\mathcal{D}) \cap \Delta_S^n(\mathcal{D})|}{q^n} \geq \left(\frac{\Delta_P^n(\mathcal{D})}{q^n}\right)^2$, then the existence of a fix-free extension $\mathcal{C} \supseteq \mathcal{D}$ which fits to $(\alpha_1, \ldots, \alpha_n)$ follows by (i) and (1.26) for $x = \frac{\Delta_P^n(\mathcal{D})}{q^n}$.
q.e.d

The proof of the lemma shows, that the condition in (i) imply the condition in (ii).

There is another difference between fix-free codes and prefix-free codes which was mentioned in [7]:

We call a finite sequence $\vec{l_n} := (l_1, \ldots l_n) \in \mathbb{N}^n$ a lengths sequence, if $l_1 \leq l_2 \leq \ldots \leq l_n$. A set $\mathcal{C} \subseteq \mathcal{A}^+$ fits to the lengths sequence $\vec{l_n}$ if $l_1, \ldots, l_n$ are the lengths of the words in $\mathcal{C}$. This means there exists a word $c_i \in \mathcal{C}$ with $|c_i| = l_i$ for every $1 \leq i \leq n$ and $\mathcal{C} = \{c_1, \ldots, c_n\}$. If $\alpha_l := |\mathcal{C} \cap \mathcal{A}^l|$ for all $l \in \mathbb{N}$, then α_l is the number of occurrence of l in the lengths sequence $\vec{l_n}$. We call the sequence $(\alpha_l)_{l \in \mathbb{N}}$ the sequence which corresponds to the lengths sequence $\vec{l_n}$. It follows that:

$$\sum_{i=1}^{n} q^{-l_i} = \sum_{l=1}^{l_n} \alpha_l q^{-l} \quad \text{and} \quad \alpha_l = 0 \ \text{ for all } \ l > l_n$$

Let $\vec{l_n} = (l_1, \ldots, l_n), \vec{l'_n} = (l'_1, \ldots, l'_n)$ be lengths sequences. We write $\vec{l'_n} \geq \vec{l_n}$ if $l'_i \geq l_i$ for all $1 \leq i \leq n$.

Let $\mathcal{C} \subseteq \mathcal{A}^+$ be a prefix-free code with lengths sequence $\vec{l_n}$ and $\vec{l'_n}$ be another lengths sequence with $\vec{l'_n} \geq \vec{l_n}$. Since $\mathcal{C}$ is prefix-free, it follows:

$$1 \geq S(\mathcal{C}) = \sum_{i=1}^{n} q^{-l_i} \geq \sum_{i=1}^{n} q^{-l'_i} \, .$$

¿From Theorem 2 follows, that there exists a prefix-free code $\mathcal{C}'$ which fits to the length sequence $\vec{l_n}$. Furthermore $\mathcal{C}'$ can be chosen in such a way, that $\mathcal{C}'$ lies in the prefix-shadow $\mathcal{C}\mathcal{A}^*$ of $\mathcal{C}$. Let $\mathcal{C} = \{c_1, \ldots, c_n\}$ with $|c_i| = l_i$ for all $1 \leq i \leq n$. We obtain $\mathcal{C}'$ by replacing of every c_i by a word $c'_i \in c_i \mathcal{A}^{l'_i - l_i} \subseteq \Delta_P^{l'_i}(\mathcal{C})$. Then the set $C' := \{c'_1, \ldots, c'_n\}$ is a prefix-free code which fits to $\vec{l'_n}$ and $\mathcal{C}' \subseteq \mathcal{C}\mathcal{A}^* = \bigcup_{l=0}^{\infty} \Delta_P^l(\mathcal{C})$.

Proposition 21 *Let $\mathcal{C} \subseteq \mathcal{A}^+$ a prefix-free code which fits to a length sequence $\vec{l_n}$ and let $\vec{l'_n}$ another lengths sequence with $\vec{l'_n} \geq \vec{l_n}$.*

(i) There exists a prefix-free Code $\mathcal{C}'$ which fits to $\vec{l'_n}$.

(ii) $\mathcal{C}'$ can be chosen in such a way that $\mathcal{C}' \subseteq \mathcal{C}\mathcal{A}^*$.

The question rises, wether the proposition above is also true for fix-free codes? The next example shows, that this is not the case.

Example 9 Let $\mathcal{A} = \{0,1\}$, $\mathcal{C} := \{0, 11, 101, 1001\}$ and $\vec{l'}_n := (1,2,4,4)$. $\mathcal{C}$ is a fix-free code with lengths sequence $\vec{l_4} = (1,2,3,4)$. We have $\vec{l'}_n \geq \vec{l_n}$. Let us assume that there exists a fix-free code $\mathcal{C}' = \{c_1', c_2', c_3', c_4'\}$ with $c_i' = l'_i$ for all $1 \leq i \leq 4$.

Case $c_1' = 0$ *:* Since $\mathcal{C}'$ is fix-free it follows that $c_2' = 11$. Then it is easy to verify that the 1001 is the only word in $\mathcal{A}^4$ which is not in the bifix-shadow of $\{c_1', c_2'\}$. This is a contradiction, because $\mathcal{C}'$ contains two words of length 4.

Case $c_1 = 1$ *:*It follows that $c_2' = 00$. Then 0110 is only word of length 4 which is not in the bifix-shadow of $\{c_1, c_2\}$. This is once again a contradiction.

This shows that there does not exists a fix-free code which fits to $\vec{l_4'}$. For the Kraftsum of $\vec{l_n}$ we obtain:

$$\sum_{i=1}^{4} 2^{-l_i} = \frac{13}{16} > \frac{3}{4} \quad \text{and} \quad \sum_{i=1}^{4} 2^{-l_i'} = \frac{7}{8} > \frac{3}{4}.$$

If the $\frac{3}{4}$-conjecture holds, then obviously Proposition 21 (i) holds fix-free codes with Kraftsum smaller than or equal to $\frac{3}{4}$. In general Proposition 21 (i) holds for fix-free codes with Kraftsum smaller than or equal to γ, if (1.17) holds for γ. On the other hand, if we assume that the $\frac{3}{4}$-conjecture hold , the next example shows that Proposition 21 (ii) does not hold for fix-free codes with Kraftsum smaller than or equal to $\frac{3}{4}$.

Example 10 Let $\mathcal{A} = \{0,1\}$ and $\mathcal{C} := \{011, 110, 010, 1001\}$. Then $\mathcal{C}$ is a fix-free code with lengths $\vec{l_4} = (3,3,3,4)$ and Kraftsum $\frac{7}{16} < \frac{3}{4}$. Let $\vec{l'}_4 := (3,3,3,5)$, then $\vec{l_n} \leq \vec{l'}_n$. Every word in $\Delta_B^5(1001) = \{11001, 01001, 10010, 10011\}$ has at least one word in $\{011, 110, 010\} = \mathcal{D} \cap A^3$ as a suffix or as a prefix. It follows that there does not exist a fix-free Code $\mathcal{C}' \subseteq \mathcal{C}\mathcal{A}^*$ with lengths sequence $\vec{l'}_4$. On the other hand there exists a fix-free Code with lengths sequence $\vec{l'}_4$. For example $\mathcal{C}' := \{011, 110, 010, 10001\}$.

Chapter 2

The $\frac{3}{4}$-conjecture for q-ary fix-free codes

This chapter is about the cases, where the $\frac{3}{4}$-conjecture can be shown for an arbitrary finite alphabet $\mathcal{A}$. We show first two theorems from Ahlswede, Balkenhol and Khachatrian [5] and Harada and Kobayashi [6] which stated, that the $\frac{3}{4}$-conjecture holds for sequences with $2k \leq \inf\{l \,|\, \alpha_l \neq 0\,, l > k\}$ for all $k \in \mathbb{N}$ and that the $\frac{3}{4}$-conjecture holds for two level fix-free codes. Finally we give a generalization of a theorem from Kukorelly and Zeger [10], which was shown for the binary case originally. This theorem shows, that the $\frac{3}{4}$-conjecture holds for finite codes, if the number of codewords on each level, expect the maximal level, is bounded by term which depends on the minimal level. The generalization of this theorem for q-ary alphabets is one of the new results in this survey.

The next theorem shows that $\frac{3}{4}$-conjecture holds for sequences with property $P^*_{\frac{3}{4}}$. It was first shown by Ahlswede, Balkenhol and Khachatrian in [5] for binary codes and finite sequences. In [6] Harada and Kobayashi generalized the theorem to the form given below for arbitrary finite alphabets and infinite sequences.

Theorem 4 (Ahlswede, Balkenhol and Khachatrian) *Let $|\mathcal{A}| = q \geq 2$ and $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative integers. If the sequence has the properties*

$$\sum_{l=1}^{\infty} \alpha_l q^{-l} \leq \frac{3}{4} \quad \textit{and} \quad 2k \leq \inf\{l \,|\, \alpha_l \neq 0\,, l > k\} \quad \textit{for all} \quad k \in \mathbb{N} \quad \textit{with} \quad \alpha_k \neq 0\,,$$

then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^*$, which fits to $(\alpha_n)_{n\in\mathbb{N}}$.

Furthermore from the proof of the theorem follows, that the class of fix-free sets is a $P^*_{\frac{3}{4}}$-simple extension class.

Proof: (by induction)

(i) Let $k_1 > 0$ the smallest number with $\alpha_{k_1} \neq 0$. It follows that:

$$\frac{3}{4} \geq \sum_{l=1}^{k_1} \alpha_l q^{-l} = \alpha_{k_1} q^{-k_1} \Rightarrow |\mathcal{A}^{k_1}| \geq \frac{3}{4} q^{k_1} = \alpha_{k_1}$$

Therefore we can choose a set $\mathcal{C}_1 \subseteq \mathcal{A}^{k_1}$ with $|\mathcal{C}_1| = \alpha_{k_1}$. Obviously $\mathcal{C}_1$ is fix-free code which fits to $(\alpha_1, \ldots, \alpha_{k_1})$.

(ii) $\mathbf{k_i} \rightarrow \mathbf{k_{i+1}}$**:**
Let $k_i \in \mathbb{N}$ such that $\alpha_{k_i} \neq 0$ and $\mathcal{C}_i$ be a fix-free code which fits to $(\alpha_1, \ldots, \alpha_{k_i})$. Let $k_{i+1} := \inf\{\, l \in \mathbb{N} \,|\, \alpha_l > 0, l > k_i\} \leq \infty$. If $k_{i+1} = \infty$, the set $\mathcal{C} := \mathcal{C}_i$ is a fix-free code which fits to $(\alpha_n)_{n \in \mathbb{N}}$. Therefore let us suppose that $k_{i+1} < \infty$. By the conditions of the theorem follows $2k_i \leq k_{i+1}$. Therefore we obtain $|x| + |y| \leq k_{i+1}$ for all $x, y \in \mathcal{C}_i$. From Lemma 8 (iii) follows:

$$|I_{k_{i+1}}(x, y)| = q^{k_{i+1} - |x| - |y|} \text{ for all } x, y \in \mathcal{C}_i\,.$$

While $|x| < k_{i+1}$ for all $x \in \mathcal{C}_i$, from Lemma 8 follows:

$$\begin{aligned} |\Delta_P^{k_{i+1}}(\mathcal{C}_i) \cap \Delta_S^{k_{i+1}}(\mathcal{C}_i)| &= \sum_{x,y \in \mathcal{C}_i} |I_{k_{i+1}}(x, y)| = \sum_{x,y \in \mathcal{C}_i} q^{k_{i+1} - |x| - |y|} \\ &= q^{k_{i+1}} \sum_{l_1, l_2 = 1}^{k_i} \alpha_{l_1} \alpha_{l_2} q^{-l_1 - l_2} = q^{k_{i+1}} \Big(\sum_{l=1}^{k_i} \alpha_l q^{-l} \Big)^2. \end{aligned}$$

Since $|\Delta_P^{k_{i+1}}(\mathcal{C}_i)| = \sum_{x \in \mathcal{C}_1} q^{k_{i+1} - |x|} = q^{k_{i+1}} \cdot \sum_{l=1}^{k_i} \alpha_l q^{-l}$, we obtain:

$$\frac{|\Delta_P^{k_{i+1}}(\mathcal{C}_i) \cap \Delta_S^{k_{i+1}}(\mathcal{C}_i)|}{q^{k_{i+1}}} = \Big(\frac{|\Delta_P^{k_{i+1}}(\mathcal{C}_i)|}{q^{k_{i+1}}} \Big)^2.$$

Furthermore we have $\sum_{l=1}^{k_{i+1}} \alpha_l q^{-l} \leq \frac{3}{4}$. This shows, that the conditions of Lemma 20 (ii) hold. Therefore it follows that there exists a fix-free extension $\mathcal{C}_{i+1}$ of $\mathcal{C}_i$ which fits to $(\alpha_1, \ldots, \alpha_{k_i})$.

(iii) If there exists $i \in \mathbb{N}$ such that $k_i = \infty$, then $\mathcal{C} := C_i$ is a fix-free code which fits to $(\alpha_n)_{n\in\mathbb{N}}$. If for every $l \in \mathbb{N}$ there exists a $k > l$ with $\alpha_k \neq 0$, then the procedure above doesn't stop. In this case the set $\mathcal{C} := \bigcup_{i=1}^{\infty} \mathcal{C}_i$ is a fix-free code which fits to $(\alpha_n)_{n\in\mathbb{N}}$, because $\mathcal{C}_i \uparrow \mathcal{C}$. **q.e.d**

We have shown in the last section of Chapter 1, that it is in general not possible to obtain a fix-free code $\mathcal{C}$ which fits to a sequence $(\alpha_1, \dots, \alpha_n)$ with Kraftsum smaller or equal $\frac{3}{4}$, by the following procedure. Choose a set $C_1 \subseteq \mathcal{A}^1$ with $|C_1| = \alpha_1$. Then extend $\mathcal{C}_1$ to a fix-free set $\mathcal{C}_2$ which fits to (α_1, α_2), after this extend $\mathcal{C}_3$ to a fix-free set $\mathcal{C}_3$ which fits to $(\alpha_1, \alpha_2, \alpha_3)$ etc. . Although this works fine for the case in the theorem above, the next example shows, that this is even not possible for a two level fix-free code.

Example 11 Let $\mathcal{A} := \{0, 1\}$ and $\alpha_1 = \alpha_2 = 0, \alpha_3 = \alpha_4 = 4$, $\alpha_l = 0$ for $l > 4$. We obtain $\sum_{l=0}^{\infty} \alpha_l q^{-l} = \frac{3}{4}$. If we choose $C_1 = \{001, 101, 110, 111\}$, then $\mathcal{C}_1$ fits to $(\alpha_1, \dots, \alpha_3)$. The tabular below shows that $|\mathcal{A}^4| - |\Delta_B^4(\mathcal{C}_1)| = 3$. Therefore it is not possible to extend $\mathcal{C}_1$ to a fix-free code which fits to $(\alpha_1, \dots, \alpha_4)$.

Level 4	P	S	Level 4	P	S	Level 4	P	S
0000			0110		x	1100	x	
0001		x	0111		x	1101	x	x
0010	x		**1000**			1110	x	x
0011	x		1001		x	1111	x	x
0100			1010	x				
0101		x	1011	x				

The proof of the next theorem shows, how to choose the first level of a two level fix-free code, if the Kraftsum of the code is smaller than or equal to $\frac{3}{4}$. The theorem was shown by Harada and Kobayashi in [6] and shows that the $\frac{3}{4}$-conjecture holds for two level fix-free codes.

Defenition 2 *Let $q \geq 2$ and $\mathcal{A} = \{0, \ldots, q-1\}$. We define the map $num_q : \mathcal{A}^+ \to \mathbb{N}$ as:*

$$num_q(x) = \sum_{l=0}^{|x|-1} x_{|x|-l} q^l \ \ for \ \ x = x_1 \ldots x_{|x|} \in \mathcal{A}^+ \,.$$

In the following we identify a finite alphabet $\mathcal{A}$ with $\{0, \ldots, q-\}$, if $|\mathcal{A}| = q \geq 2$. Obviously the function $num_{q|_{\mathcal{A}^l}}$ is a one-to-one map onto $\{0, 1, \ldots, q^l - 1\}$ for every $l \in \mathbb{N}$.

Proposition 22 *Let $|\mathcal{A}| = q \geq 2$ and $x, y \in \mathcal{A}^+$ with $|x| \leq |y|$.*

(i): x is a suffix of $y \Leftrightarrow num_q(y) \bmod q^{|x|} = num_q(x)$

(ii): x is prefix of $y \Leftrightarrow num_q(x) \cdot q^{|y|-|x|} \leq num_q(y) < (num_q(x) + 1) q^{|y|-|x|}$.

Theorem 5 (Harada and Kobayashi) *Let $q \geq 2, \mathcal{A} = \{0, \ldots, q-1\}$, $m < n$ and $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of of nonnegative integers with $\alpha_l = 0$ for all $l \notin \{m, n\}$. If $\alpha_m q^{-m} + \alpha_n q^{-n} \leq \frac{3}{4}$, then*

(i): there exists a fix-free Code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits to $(\alpha_n)_{n \in \mathbb{N}}$.

(ii): If we choose $C_1 = \{x \in \mathcal{A}^m \,|\, 0 \leq num_q(x) \leq \alpha_m - 1\}$, then there exists a fix-free extension of $\mathcal{C}_1$ which fits to $(\alpha_n)_{n \in \mathbb{N}}$.

Proof: From $\frac{3}{4} \geq \alpha_m q^{-m} + \alpha_n q^{-n}$ follows that $\alpha_m \leq q^m = |\mathcal{A}^m|$. Thus we can define :

$$\mathcal{C}_1 := \{x \in \mathcal{A}^m \,|\, 0 \leq num_q(x) < \alpha_m\} \subseteq \mathcal{A}^m.$$

If we take in account that $num_{q|_{\mathcal{A}^m}}$ is a one-to-one map onto $\{0, \ldots, q^m - 1\}$, we obtain $|\mathcal{C}_1| = \alpha_m$. Let $s := \lfloor \frac{\alpha_m q^{n-m}}{q^m} \rfloor$. We define:

$$\begin{aligned} T_t \ &:= \ \{y \in \mathcal{A}^n \,|\, (t-1) q^m \leq num_q(y) < t q^m\} \ \text{ for all } \ 1 \leq t \leq s \ \text{ and} \\ T_{s+1} \ &:= \ \{y \in \mathcal{A}^n \,|\, s \cdot q^m \leq num_q(y) < \alpha_m q^{n-m}\} \,. \end{aligned}$$

While $\alpha_m q^{n-m} \leq \frac{3}{4}q^n \leq |\mathcal{A}^n|$ and since $num_{q|_{\mathcal{A}^n}}$ is a one-to-one map onto $\{0, \ldots, q^n - 1\}$, it follows that:

$$\begin{array}{rcl} |T_t| & = & q^m \text{ for all } 1 \leq t \leq s \text{ and} \\ |T_{s+1}| & = & \alpha_m q^{n-m} - q^m \lfloor \frac{\alpha_m q^{n-m}}{q^m} \rfloor \leq q^m \,. \end{array}$$

By Proposition 22 (ii) follows:

$$\bigcup_{t=1}^{s+1} T_t = \{ y \in \mathcal{A}^n \mid 0 \leq num_q(y) < \alpha_m \} = \Delta_P^n(\mathcal{C}_1)\,.$$

The T_i's are pairwise disjoint. Therefore $T_1, \ldots T_s, T_{s+1}$ is a partition of $\Delta_P^n(\mathcal{C}_1)$ Because of Proposition 22 (i) we we obtain for every $\mathcal{X} \subseteq A^m$ and $1 \leq t \leq q^{n-m}$:

$$\begin{array}{rl} & \{ y \in \mathcal{A}^n \mid (t-1)q^m \leq num_q(y) < tq^m \} \cap \Delta_S^n(\mathcal{X}) \\ = & \{ y \in \mathcal{A}^n \mid x \in \mathcal{X}, num_q(y) = (t-1)q^m + num_q(x) \}\,. \end{array}$$

¿From the definition of $\mathcal{C}_1, T_1, \ldots, T_{s+1}$ and $|T_{s+1}| < q^m$ follows:

$$T_t \cap \Delta_S^n(\mathcal{C}_1) = \{ y \in \mathcal{A}^n \mid (t-1)q^m \leq num_q(y) < (t-1)q^m + \alpha_m \} \text{ for } 1 \leq t \leq s\,.$$

$$T_{s+1} \cap \Delta_S^n(\mathcal{C}_1) = \begin{cases} \{ y \in \mathcal{A}^n \mid s \cdot q^m \leq num_q(y) < s \cdot q^m + \alpha_m \} & \text{for } |T_{s+1}| \geq \alpha_m \\ T_{s+1} & \text{for } |T_{s+1}| < \alpha_m \end{cases} \tag{2.1}$$

Since the T_i's are a partition of $\Delta_P^n(\mathcal{C}_1)$, we have

$$|\Delta_P^n(\mathcal{C}_1) \cap \Delta_S^n(\mathcal{C}_1)| = \sum_{i=1}^{s+1} |T_i \cap \Delta_S^n(\mathcal{C}_1)|\,.$$

By (2.1) follows:

$$\begin{array}{rcl} |\Delta_P^n(\mathcal{C}_1) \cap \Delta_S^n(\mathcal{C}_1)| & = & s \cdot \alpha_m + |T_{s+1} \cap \Delta_S^n(\mathcal{C}_1)| \\ & = & \lfloor \frac{\alpha_m q^{n-m}}{q^m} \rfloor \alpha_m + \begin{cases} \alpha_m & \text{if } |T_{s+1}| \geq \alpha_m \\ \alpha_m q^{n-m} - q^m \lfloor \frac{\alpha_m q^{n-m}}{q^m} \rfloor & \text{if } |T_{s+1}| < \alpha_m \end{cases} \end{array} \tag{2.2}$$

While $\frac{3}{4} \geq \alpha_m q^{-m} + \alpha_n q^{-n}$, from Lemma 20 (ii) follows, that it is sufficient for the existence of a set $\mathcal{C}_2 \subseteq \mathcal{A}^n$ with $|\mathcal{C}_2| = \alpha_n$ and $\mathcal{C} := \mathcal{C}_1 \cup \mathcal{C}_2$ is fix-free, to show that the following inequality holds.

$$|\Delta_P^n(\mathcal{C}_1) \cap \Delta_S^n(\mathcal{C}_1)| \geq \frac{|\Delta_P^n(\mathcal{C}_1)|^2}{q^n} \tag{2.3}$$

We show the above inequality by distinguishing two cases. This will finish the proof.

Case 1: $|T_{s+1}| \geq \alpha_m$

By equation (2.2) we have:

$$|\Delta_P^n(\mathcal{C}_1) \cap \Delta_S^n(\mathcal{C}_1)| = \lfloor \frac{\alpha_m q^{n-m}}{q^m} \rfloor \alpha_m + \alpha_m = \lfloor \frac{\alpha_m q^{n-m}}{q^m} + 1 \rfloor \alpha_m \geq \frac{\alpha_m^2 q^{n-m}}{q^m}\ . \tag{2.4}$$

Since $|\Delta_P^n(\mathcal{C}_1)| = \alpha_m q^{n-m}$, we have

$$\frac{|\Delta_P^n(\mathcal{C}_1)|^2}{q^n} = \frac{\alpha_m^2 q^{2n-2m}}{q^n} = \frac{\alpha_m^2 q^{n-m}}{q^m} \tag{2.5}$$

By equations (2.4) and (2.5), it follows that the desired inequality (2.3) holds.

Case 2: $|T_{s+1}| < \alpha_m$

In this case we have:

$$\begin{array}{ccll}
 & \frac{\alpha_m q^{n-m}}{q^m} & \geq & \lfloor \frac{\alpha_m q^{n-m}}{q^m} \rfloor \\
\underset{\alpha_m - q^m \leq 0}{\Leftrightarrow} & (\alpha_m - q^m)\frac{\alpha_m q^{n-m}}{q^m} & \leq & (\alpha_m - q^m)\lfloor \frac{\alpha_m q^{n-m}}{q^m} \rfloor \\
\Leftrightarrow & \frac{\alpha_m^2 q^{n-m}}{q^m} & \leq & \alpha_m \lfloor \frac{\alpha_m q^{n-m}}{q^m} \rfloor + \alpha_m q^{n-m} - q^m \lfloor \frac{\alpha_m q^{n-m}}{q^m} \rfloor \quad . \\
\underset{(2.2)}{\Rightarrow} & \frac{\alpha_m^2 q^{n-m}}{q^m} & \leq & |\Delta_P^n(\mathcal{C}_1) \cap \Delta_S^n(\mathcal{C}_1)| \\
\underset{(2.5)}{\Rightarrow} & \frac{|\Delta_P^n(\mathcal{C}_1)|^2}{q^n} & \leq & |\Delta_P^n(\mathcal{C}_1) \cap \Delta_S^n(\mathcal{C}_1)|
\end{array}$$

Thus the inequality (2.3) holds in this case, as well. **q.e.d**

In [10] Kukorelly and Zeger show the $\frac{3}{4}$-conjecture for binary codes and finite sequences, if the number of codewords on each level which is smaller than the maximal level is limited by $2^{l_{min}-2}$, where l_{min} is the first nonempty level of the code.

Theorem 6 (Kukorelly and Zeger) *Let $A = \{0,1\}$, $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative integers with $\sum_{l=1}^{\infty} \alpha_l \left(\frac{1}{2}\right)^l \leq \frac{3}{4}$, $l_{min} := \min\{l|\alpha_l > 0\}$ and $l_{max} := sup\{l \in \mathbb{N}|\alpha_l > 0\} \leq \infty$. If $l_{min} \geq 2$, $l_{max} < \infty$ and $\alpha_l \leq 2^{l_{min}-2}$ for all $l \neq l_{max}$, then there exists a fix-free code $\mathcal{C} \subseteq \{0,1\}^*$ which fits to $(\alpha_n)_{n\in\mathbb{N}}$.*

We prove a generalization of the theorem above for arbitrary finite alphabets. This is one of the new results in this survey. However the proof of the generalization is similar to the proof of the binary case given in [10], if the binary alphabet $\{0,1\}$ is replaced by $\{X,Y\}$, where $\mathcal{X},\mathcal{Y}$ is a partition of the alphabet $\mathcal{A}$ with $|\mathcal{X}| = \left\lfloor \frac{\mathcal{A}}{2} \right\rfloor$ and $|\mathcal{Y}| = \left\lceil \frac{\mathcal{A}}{2} \right\rceil$.

Theorem 7 *Let $|\mathcal{A}| = q \geq 2$, $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers with $\sum_{l=l_{min}}^{l_{max}} \alpha_l q^{-l} \leq \frac{3}{4}$ and $l_{min} := \min\{l\,|\alpha_l \geq 0\}$, $l_{max} := sup\{l\,|\alpha_l \geq\} \leq \infty$. If $l_{min} \geq 2$, $l_{max} < \infty$ and $\alpha_l \leq q^{l_{min}-2}\lfloor\frac{q}{2}\rfloor^2\lceil\frac{q}{2}\rceil^{l-l_{min}}$ for all $l \neq l_{max}$, then there exists a fix-free Code $\mathcal{C} \subseteq \mathcal{A}^*$ which fits to $(\alpha_l)_{l\in\mathbb{N}}$.*

Theorem 6 follows from Theorem 7 for $q = 2$.

Proof: If $l_{max} \leq l_{min} + 1$ then we need only a one level or two level code. In this case the theorem follows from Theorem 5. Thus we can assume that $l_{max} \geq l_{min} + 2$. In the proof we first construct a fix-free Code $\mathcal{C}_0$ such that $|\mathcal{C}_0 \cap \mathcal{A}^l| = q^{l_{min}-2}\lfloor\frac{q}{2}\rfloor^2\lceil\frac{q}{2}\rceil^{l-l_{min}}$ for all $l_{min} \leq l < l_{max}$ and $\sum_{l=l_{min}}^{l_{max}} |\mathcal{C}_0 \cap \mathcal{A}^l| q^{-l} = \frac{3}{4}$.

Then in four steps we delete $q^{l_{min}-2}\lfloor\frac{q}{2}\rfloor^2\lceil\frac{q}{2}\rceil^{l-l_{min}} - \alpha_l$ codewords from each level $l_{min} \leq l < l_{max}$, replace each of this codeword with more than $q^{l_{max}-l}$ new codewords on the l_{max}-th level, and show with Lemma 11, that this new code $\mathcal{C}$ is also fix-free. Then the Kraftsum of this new Code is bigger or equal $\frac{3}{4}$, $|\mathcal{C} \cap \mathcal{A}^l| = \alpha_l \; \forall\, l_{min} \leq l < l_{max}$ and $|\mathcal{C} \cap \mathcal{A}^{l_{max}}| \geq \alpha_{l_{max}}$. To obtain the desired Code we have only to delete some codewords on the l_{max}-th level.

Let $\mathcal{X}, \mathcal{Y}$ be a partition of the alphabet $\mathcal{A}$ into two parts with $|\mathcal{X}| = \lfloor \frac{q}{2} \rfloor$ and $|\mathcal{Y}| = \lceil \frac{q}{2} \rceil$. We define:

$$\begin{aligned}
\mathcal{B}_0 &:= \{x_1 y x_2 | x_1, x_2 \in \mathcal{X},\, y \in \mathcal{Y}^i,\, 0 \leq i \leq l_{max} - l_{min} - 1\} \\
\mathcal{D}_1 &:= \mathcal{Y}\mathcal{A}^{l_{max}-l_{min}}\mathcal{Y}\mathcal{A}^{l_{min}-2} \subseteq \mathcal{A}^{l_{max}} \\
\mathcal{D}_2 &:= \mathcal{X}\mathcal{Y}^{l_{max}-l_{min}}\mathcal{A}^{l_{min}-1} \subseteq \mathcal{A}^{l_{max}} \\
\mathcal{B} &:= \mathcal{B}_0\mathcal{A}^{l_{min}-2} \subseteq \bigcup_{l=l_{min}}^{l_{max}-1} \mathcal{A}^l \\
\mathcal{C}_0 &:= \mathcal{B} \cup D_1 \cup D_2
\end{aligned}$$

$\mathcal{B}$ is fix-free, because $\mathcal{B}_0$ is fix-free. Obviously no codeword in $\mathcal{B}$ is a prefix or suffix from a word in $\mathcal{D}_1 \cup \mathcal{D}_2$. Thus $\mathcal{C}_0$ is fix-free, as well.

We have:

$$\begin{aligned}
|\mathcal{C}_0 \cap A^l| = |\mathcal{B} \cap A^l| &= q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}} \qquad \text{for } l_{min} \leq l < l_{max}, \quad (2.6) \\
|\mathcal{D}_1| &= q^{l_{max}-2}\left\lceil\frac{q}{2}\right\rceil^2, \\
|\mathcal{D}_2| &= \left\lfloor\frac{q}{2}\right\rfloor\left\lceil\frac{q}{2}\right\rceil^{l_{max}-l_{min}} q^{l_{min}-1}.
\end{aligned}$$

It follows:

$$\begin{aligned}
S(\mathcal{D}_1) &= |\mathcal{D}_1|q^{-l_{max}} = \left(\frac{1}{q}\right)^2\left\lceil\frac{q}{2}\right\rceil^2, \\
S(\mathcal{D}_2) &= |\mathcal{D}_2|q^{-l_{max}} = \left\lfloor\frac{q}{2}\right\rfloor\left\lceil\frac{q}{2}\right\rceil^{l_{max}-l_{min}} q^{l_{min}-l_{max}-1} = \frac{1}{q}\left\lfloor\frac{q}{2}\right\rfloor\left(\frac{1}{q}\left\lceil\frac{q}{2}\right\rceil\right)^{l_{max}-l_{min}}, \\
S(\mathcal{B}) &= \sum_{l=l_{min}}^{l_{max}-1} |\mathcal{B} \cap A^l| q^{-l} = q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2 \sum_{l=l_{min}}^{l_{max}-1}\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}}\left(\frac{1}{q}\right)^l, \\
&= \left(\frac{1}{q}\left\lfloor\frac{q}{2}\right\rfloor\right)^2 \cdot \sum_{l=0}^{l_{max}-l_{min}-1}\left(\left\lceil\frac{q}{2}\right\rceil\frac{1}{q}\right)^l, \\
&= \left(\frac{1}{q}\left\lfloor\frac{q}{2}\right\rfloor\right)^2 \cdot \frac{1-\left(\lceil\frac{q}{2}\rceil\frac{1}{q}\right)^{l_{max}-l_{min}}}{1-\lceil\frac{q}{2}\rceil\frac{1}{q}}.
\end{aligned}$$

We obtain from the last equation:

$$S(\mathcal{B}) = \frac{1}{q}\left\lfloor\frac{q}{2}\right\rfloor - \frac{1}{q}\left\lfloor\frac{q}{2}\right\rfloor \left(\left\lceil\frac{q}{2}\right\rceil\frac{1}{q}\right)^{l_{max}-l_{min}} = \frac{1}{q}\left\lfloor\frac{q}{2}\right\rfloor - S(\mathcal{D}_2)\,.$$

The sets $\mathcal{B}, \mathcal{D}_1$ and $\mathcal{D}_2$ are disjoint and so together with the equations above follows:

$$S(\mathcal{C}_0) = S(\mathcal{B}) + S(\mathcal{D}_1) + S(\mathcal{D}_2) = \frac{1}{q}\left\lfloor\frac{q}{2}\right\rfloor + \left(\frac{1}{q}\right)^2\left\lceil\frac{q}{2}\right\rceil^2 . \tag{2.7}$$

We claim

$$\frac{1}{q}\left\lfloor\frac{q}{2}\right\rfloor + \left(\frac{1}{q}\right)^2\left\lceil\frac{q}{2}\right\rceil^2 \geq \frac{3}{4}\,.$$

If q is even, we have $\frac{1}{q}\lfloor\frac{q}{2}\rfloor + (\frac{1}{q})^2\lceil\frac{q}{2}\rceil^2 = \frac{1}{2} + (\frac{1}{2})^2 = \frac{3}{4}$.

If q is odd, we have $q = 2p+1$ for some $p \in \mathbb{N}$ and we obtain:

$$\begin{aligned}
& \frac{1}{q}\left\lfloor\frac{q}{2}\right\rfloor + \left(\frac{1}{q}\right)^2\left\lceil\frac{q}{2}\right\rceil^2 \geq \frac{3}{4} \\
\Leftrightarrow\quad & \frac{p}{2p+1} + \left(\frac{p+1}{2p+1}\right)^2 \geq \frac{3}{4} \\
\Leftrightarrow\quad & (2p+1)p + (p+1)^2 \geq \frac{3}{4}(2p+1)^2 \\
\Leftrightarrow\quad & 3p^2 + 3p + 1 \geq 3p^2 + 3p + \frac{3}{4}\,.
\end{aligned}$$

This holds for all $p \in \mathbb{N}$. With (2.7) follows:

$$S(\mathcal{C}_0) = \sum_{l=l_{min}}^{l_{max}} |\mathcal{C}_0 \cap \mathcal{A}^l| q^{-l} \geq \frac{3}{4}\,. \tag{2.8}$$

Let $\mathcal{E} \subseteq \mathcal{B}$ be a set which contains $q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}} - \alpha_l$ codewords of length l for each $l_{min} \leq l < l_{max}$. Furthermore let $F \subseteq \mathcal{A}^{l_{max}} - (\mathcal{D}_1 \cup \mathcal{D}_2)$ be an arbitrary set of at least

$$\sum_{l=l_{min}}^{l_{max}-1} \left(q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}} - \alpha_l\right) q^{l_{max}-l}$$

codewords. If we remove the words of $\mathcal{E}$ from $\mathcal{C}_0$ and add the words of $\mathcal{F}$ we obtain the set:

$$\mathcal{C} := (\mathcal{B} - \mathcal{E}) \cup (\mathcal{D}_1 \,\dot\cup\, \mathcal{D}_2 \,\dot\cup\, \mathcal{F}) = (\mathcal{C}_0 - \mathcal{E}) \,\dot\cup\, \mathcal{F}\,.$$

For this set we have:

$$\left|\mathcal{C} \cap A^l\right| = \alpha_l \text{ for } l_{min} \leq l < l_{max} \,.$$

$$\begin{aligned}
S(\mathcal{C}) &= S(\mathcal{C}_0) - S(\mathcal{E}) + S(\mathcal{F}) \\
&\geq S(\mathcal{C}_0) - \sum_{l=l_{min}}^{l_{max}-1} q^{-l}\left(q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}} - \alpha_l\right) \\
&\quad + q^{-l_{max}} \cdot \sum_{l=l_{min}}^{l_{max}-1}\left(q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}} - \alpha_l\right) q^{l_{max}-l} \\
&= S(\mathcal{C}_0) \geq \frac{3}{4} \,.
\end{aligned}$$

We obtain a set which fits to $(\alpha_l)_{l\in\mathbb{N}}$, by deleting some words of length l_{max} from $\mathcal{C}$, because $\sum\limits_{l=0}^{\infty} \alpha_l q^{-l} = \sum\limits_{l=l_{min}}^{l_{max}-1} \alpha_l q^{-l} \leq \frac{3}{4}$

To complete the proof we have to show, that we can choose $\mathcal{E}$ and $\mathcal{F}$ such that $\mathcal{C}$ is fix-free. To show this, we use Lemma 11, which says, that $\mathcal{C}$ is fix-free if the following two conditions holds.

(i) Each word in $\mathcal{F}$ has a prefix in $\mathcal{E}$ or no prefix in $\mathcal{C}_0$.

(ii) Each word in $\mathcal{F}$ has a suffix in $\mathcal{E}$ or no suffix in $\mathcal{C}_0$.

We construct the sets $\mathcal{F}$ and $\mathcal{E}$ in three steps:

1. For each $l_{min} \leq l \leq l_{max} - l_{min} + 1$ we include in $\mathcal{E}_1 \subseteq \mathcal{B}$ all $q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}} - \alpha_l$ words of the form:

$$x_1 y x_2 w \,, \quad \text{where } \; x_1, x_2 \in \mathcal{X},\, y \in \mathcal{Y}^{l-l_{min}},\, w \in \mathcal{A}^{l_{min}-2}.$$

   For each of these words, we include in $\mathcal{F}_1 \subseteq \mathcal{A}^{l_{max}} - (\mathcal{D}_1 \cup \mathcal{D}_2)$:

   a) the $\lceil\frac{q}{2}\rceil q^{l_{max}-l-1}$ words of the sets:

$$x_1 y x_2 w \mathcal{A}^{l_{max}-l_{min}-l+1} \mathcal{Y} \mathcal{A}^{l_{min}-2} \subseteq \mathcal{A}^{l_{max}} \tag{2.9}$$

   Each of these words has a prefix in $\mathcal{E}_1$, but they have no suffix in $\mathcal{B}$, because for every word in $\mathcal{E}_1$ the $(l_{min}-1)$-th letter from the left-hand side is an element of $\mathcal{Y}$. Furthermore each word of $\mathcal{F}_1$ has a prefix of $x_1 y x_2 \in \mathcal{C}_0$ Thus $\mathcal{F}_1$ is disjoint from $\mathcal{D}_1 \cup \mathcal{D}_2$, since $\mathcal{C}$ is prefix-free.

b) choose $\lfloor \frac{q}{2} \rfloor q^{l_{max}-l-1}$ arbitrary words of the sets:

$$\mathcal{Y}\mathcal{A}^{l_{max}-l-1}x_1yx_2w \subseteq \mathcal{A}^{l_{max}} \tag{2.10}$$

Each of these words have a suffix in $\mathcal{E}_1$, but they have no prefix in $\mathcal{B}$, because they begin with a letter in $\mathcal{Y}$. Since $\mathcal{C}_0$ is suffix-free, none of these words are in $\mathcal{D}_1 \cup \mathcal{D}_2$ and of course also disjoint from the other part of $\mathcal{F}_1$.

Thus for the sets $\mathcal{E}_1, \mathcal{F}_1$ the above conditions of Lemma 11 holds and we obtain:

$$|\mathcal{F}_1| = \sum_{l=l_{min}}^{l_{max}-l_{min}+1} q^{l_{max}-l}\left(q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}} - \alpha_l\right). \tag{2.11}$$

2. For each $l_{max}-l_{min}+2 \leq l < l_{max}$ and $\alpha_l \geq q^{l_{min}-3}\left\lfloor\frac{q}{2}\right\rfloor^3\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}}$ we include in $\mathcal{E}_2 \subseteq \mathcal{B}$ any $q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}} - \alpha_l$ words of the form:

$$\begin{aligned}&x_1yx_2w \in \mathcal{B} \text{ with } x_1, x_2 \in \mathcal{X},\ y \in \mathcal{Y}^{l-l_{min}} \text{ and}\\ &w \in \mathcal{A}^{l_{max}-l-1}\mathcal{Y}\mathcal{A}^{l-(l_{max}-l_{min}+2)} \subseteq \mathcal{A}^{l_{min}-2}\end{aligned} \tag{2.12}$$

The letters at the $(l_{max}-l_{min}+2)$-th position of these words of $\mathcal{B}$ are in $\mathcal{Y}$. For each possible l there are $q^{l_{min}-3}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}+1}$ such words and from the condition for α_l follows:

$$\begin{aligned}q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}} - \alpha_l &\leq q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}} - q^{l_{min}-3}\left\lfloor\frac{q}{2}\right\rfloor^3\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}}\\ &= q^{l_{min}-3}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}}\cdot\left(q-\left\lfloor\frac{q}{2}\right\rfloor\right)\\ &= q^{l_{min}-3}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}+1}\end{aligned}$$

Therefore we can include enough words in $\mathcal{E}_2$.
For each of this words we include in $\mathcal{F}_2 \subseteq \mathcal{A}^{l_{max}} - (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{F}_1)$:

a) the $\left\lceil\frac{q}{2}\right\rceil q^{l_{max}-l-1}$ words of the set:

$$\mathcal{Y}\mathcal{A}^{l_{max}-l-1}x_1yx_2w \subseteq \mathcal{A}^{l_{max}} \tag{2.13}$$

These words have a suffix in $\mathcal{E}_2$,but they have no prefix in $\mathcal{B}$, because they begin with a letter in $\mathcal{Y}$. Moreover they are neither contained in $\mathcal{F}_1$ nor they are contained in $\mathcal{D}_1 \cup \mathcal{D}_2$, because $\mathcal{C}_0$ is suffix-free.

b) choose any $\left\lfloor \frac{q}{2} \right\rfloor q^{l_{max}-l-1}$ from the set:

$$x_1 y x_2 w \mathcal{A}^{l_{max}-l} \subseteq \mathcal{A}^{l_{max}} \tag{2.14}$$

These words have a prefix in $\mathcal{E}_2$, but they have no suffix in $\mathcal{B}$, because they have a letter at the $(l_{max}-l_{min}+2)$-th position which is an element of $\mathcal{Y}$ and therefore ends with with a word in $\mathcal{Y}\mathcal{A}^{l_{min}-2}$, whereas all codewords in $\mathcal{B}$ ends with a word in $\mathcal{X}\mathcal{A}^{l_{min}-2}$. Furthermore they are not contained in $\mathcal{D}_1 \cup \mathcal{D}_2$, because $\mathcal{C}_0$ is prefix-free and obviously they are also not contained in $\mathcal{F}_1$.

Thus for the sets $\mathcal{E}_2$ and $\mathcal{F}_2$ the conditions of the lemma holds. For every possible l the number of codewords in $\mathcal{F}_2$ is:

$$\begin{aligned} & \left(q^{l_{min}-2}\left\lfloor \frac{q}{2} \right\rfloor^2 \left\lceil \frac{q}{2} \right\rceil^{l-l_{min}} - \alpha_l\right) \cdot \left(\left\lceil \frac{q}{2} \right\rceil q^{l_{max}-l-1} + \left\lfloor \frac{q}{2} \right\rfloor q^{l_{max}-l-1}\right) \\ = & \left(q^{l_{min}-2}\left\lfloor \frac{q}{2} \right\rfloor^2 \left\lceil \frac{q}{2} \right\rceil^{l-l_{min}} - \alpha_l\right) \cdot q^{l_{max}-l}. \end{aligned}$$

¿From this follows with $\beta_l := q^{l_{min}-3}\left\lfloor \frac{q}{2} \right\rfloor^3 \left\lceil \frac{q}{2} \right\rceil^{l-l_{min}}$:

$$|\mathcal{F}_2| = \sum_{\substack{l=l_{max}-l_{min}+2 \\ \alpha_l \geq \beta_l}}^{l_{max}-1} \left(q^{l_{min}-2}\left\lfloor \frac{q}{2} \right\rfloor^2 \left\lceil \frac{q}{2} \right\rceil^{l-l_{min}} - \alpha_l\right) \cdot q^{l_{max}-l}. \tag{2.15}$$

3. For each $l_{max}-l_{min}+2 \leq l < l_{max}$ and $\alpha_l < q^{l_{min}-3}\left\lfloor \frac{q}{2} \right\rfloor^3 \left\lceil \frac{q}{2} \right\rceil^{l-l_{min}}$ we include in $\mathcal{E}_3 \subseteq \mathcal{B}$ $\quad q^{l_{min}-3}\left\lfloor \frac{q}{2} \right\rfloor^2 \left\lceil \frac{q}{2} \right\rceil^{l-l_{min}+1}$ codewords of the form:

$$\begin{aligned} & x_1 y x_2 w \in \mathcal{B} \text{ with } x_1, x_2 \in \mathcal{X},\ y \in \mathcal{Y}^{l-l_{min}} \text{ and} \\ & w \in \mathcal{A}^{l_{max}-l-1}\mathcal{Y}\mathcal{A}^{l-(l_{max}-l_{min}+2)} \subseteq \mathcal{A}^{l_{min}-2} \end{aligned} \tag{2.16}$$

All these words are contained in $\mathcal{B} \cap \mathcal{A}^l$ and therefore the letter at the $(l_{max}-l_{min}+2)$-th position is in $\mathcal{Y}$

Furthermore we include in $\mathcal{E}_3$ any $q^{l_{min}-3}\left\lfloor \frac{q}{2} \right\rfloor^3 \left\lceil \frac{q}{2} \right\rceil^{l-l_{min}} - \alpha_l$ words of the form:

$$\begin{aligned} & x_1 y x_2 w \in \mathcal{B} \text{ with } x_1, x_2 \in \mathcal{X},\ y \in \mathcal{Y}^{l-l_{min}} \text{ and} \\ & w \in \mathcal{A}^{l_{max}-l-1}\mathcal{X}\mathcal{A}^{l-(l_{max}-l_{min}+2)} \subseteq \mathcal{A}^{l_{min}-2} \end{aligned} \tag{2.17}$$

Each of these words have at the $(l_{max}-l_{min}+2)$-th position a letter in $\mathcal{X}$.

For each possible l the number of codewords in $\mathcal{E}_3$ of length l is:

$$\begin{aligned}
& \left(q^{l_{min}-3}\left\lfloor\frac{q}{2}\right\rfloor^3\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}} - \alpha_l\right) + q^{l_{min}-3}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}+1} \\
= \;& q^{l_{min}-3}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}}\left(\left\lfloor\frac{q}{2}\right\rfloor + \left\lceil\frac{q}{2}\right\rceil\right) - \alpha_l \\
= \;& q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}} - \alpha_l
\end{aligned}$$

codewords in $\mathcal{E}_3$.

For each word in $\mathcal{E}_3$ of the form (2.16) or (2.17) we include in $\mathcal{F}_3$:

a) the $\left\lceil\frac{q}{2}\right\rceil q^{l_{max}-l-1}$ words of the set:

$$\mathcal{Y}\mathcal{A}^{l_{max}-l-1}x_1yx_2w \subseteq \mathcal{A}^{l_{max}} \tag{2.18}$$

These words have a suffix in $\mathcal{E}_3$, but do not have a prefix in $\mathcal{B}$, because the first letter is an element of $\mathcal{Y}$. Obviously they are not contained in $\mathcal{F}_1 \cup \mathcal{F}_2$ and they are also not contained in $\mathcal{D}_1 \cup \mathcal{D}_2$, because $\mathcal{C}_0$ is suffix-free.

For every word in $\mathcal{E}_3$ of the form (2.16) we include in $\mathcal{F}_3$:

b) the $q^{l_{max}-l}$ words of the set:

$$x_1yx_2w\mathcal{A}^{l_{max}-l} \subseteq \mathcal{A}^{l_{max}} \tag{2.19}$$

These words have a prefix in $\mathcal{E}_3$, but do not have a suffix in $\mathcal{B}$, because they have a letter at the $(l_{max} - l_{min} + 2)$-th position which is in $\mathcal{Y}$ and therefore they have a suffix in $\mathcal{Y}\mathcal{A}^{l_{min}-2}$ whereas all codewords in $\mathcal{B}$ have a suffix in $\mathcal{X}\mathcal{A}^{l_{min}-2}$. They are not in $\mathcal{D}_1 \cup \mathcal{D}_2$, because $\mathcal{C}_0$ is prefix-free and obviously they are also not contained in $\mathcal{F}_1 \cup \mathcal{F}_2$.

Therefore $\mathcal{E}_3, \mathcal{F}_3$ fulfill the condition of the lemma and
$\mathcal{F}_3 \subseteq \mathcal{A}^{l_{max}} - (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2)$.

For every possible l the number of codewords of length l in $\mathcal{F}_3$ is equal to

$$\begin{aligned}
& q^{l_{min}-3}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}+1}\cdot\left(\left\lceil\frac{q}{2}\right\rceil q^{l_{max}-l-1}+q^{l_{max}-l}\right) \\
& +\left(q^{l_{min}-3}\left\lfloor\frac{q}{2}\right\rfloor^3\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}}-\alpha_l\right)\cdot\left\lceil\frac{q}{2}\right\rceil q^{l_{max}-l-1} \\
= \;& q^{l_{max}-l}\cdot\left(q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}}\cdot\left(\frac{1}{q^2}\left\lceil\frac{q}{2}\right\rceil^2+\frac{1}{q}\left\lceil\frac{q}{2}\right\rceil+\frac{1}{q^2}\left\lceil\frac{q}{2}\right\rceil\left\lfloor\frac{q}{2}\right\rfloor\right)-\alpha_l\frac{1}{q}\left\lceil\frac{q}{2}\right\rceil\right) \\
= \;& q^{l_{max}-l}\cdot\left(q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}}\cdot\left(\frac{2}{q}\left\lceil\frac{q}{2}\right\rceil\right)-\alpha_l\frac{1}{q}\left\lceil\frac{q}{2}\right\rceil\right) \\
\geq \;& q^{l_{max}-l}\cdot\left(q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}}-\alpha_l\right).
\end{aligned}$$

With $\beta_l := q^{l_{min}-3}\left\lfloor\frac{q}{2}\right\rfloor^3\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}}$ follows:

$$|\mathcal{F}_3| \geq \sum_{\substack{l=l_{max}-l_{min}+2 \\ \alpha_l<\beta_l}}^{l_{max}-1}\left(q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}}-\alpha_l\right)\cdot q^{l_{max}-l}. \tag{2.20}$$

Let $\mathcal{E} := \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \subseteq \mathcal{B}$ and $\mathcal{F} := \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \subseteq \mathcal{A}^{l_{max}} - (\mathcal{D}_1 \cup \mathcal{D}_2)$ then from Lemma **??** it follows that $\mathcal{C} := (\mathcal{C}_0 - \mathcal{E}) \cup \mathcal{F}$ is fix-free. Moreover we have $|\mathcal{C} \cap \mathcal{A}^l| = |(\mathcal{B} \cap \mathcal{A}^l)| - |(\mathcal{E} \cap \mathcal{A}^l)| = \alpha_l$ for all $l_{min} \leq l < l_{max}$. Since $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ are disjoint, by (2.11), (2.15) and (2.20) follows:

$$|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| \geq \sum_{l=l_{min}}^{l_{max}-1}\left(q^{l_{min}-2}\left\lfloor\frac{q}{2}\right\rfloor^2\left\lceil\frac{q}{2}\right\rceil^{l-l_{min}}-\alpha_l\right)\cdot q^{l_{max}-l}$$

As described above, we obtain $S(\mathcal{C}) \geq S(\mathcal{C}_0) \geq \frac{3}{4}$ and because of $\sum_{l=l_{min}}^{l_{max}} \alpha_l \cdot q^{-l} \leq \frac{3}{4}$ we obtain a fix-free code which fits $(\alpha_l)_{l\in\mathbb{N}}$ by deleting some codewords of length l_{max} from $\mathcal{C}$. **q.e.d**

Chapter 3

The de Bruijn digraphs $\mathcal{B}_q(n)$

In this chapter we examine the existence of certain cycles and regular subgraphs of the de Bruijn digraphs. In Chapter 4 we will generate fix-free codes with $k \cdot L$ codewords on the first nonempty level with the help of k-regular subgraphs with L vertices of de Bruijn digraps. Therefore it is important to know for which numbers L of vertices such subgraphs exist. The de Bruijn digraph of span n over an alphabet $\mathcal{A}$ contains all $\mathcal{A}$-words of length n as its vertices and for every word $w \in \mathcal{A}^n$ the successors of w are given by the words which are contained in the set $\mathcal{A}^{-1}w\mathcal{A}$. de Bruijn digraphs were first constructed by de Bruijn [29](1946) and independently by Good [30](1946), while examining the existence of binary cyclic sequences of length 2^n containing 2^n different subwords of length n. Such sequences are called a (binary) de Bruijn sequence and they correspond with Hamilton circuits of the de Bruijn digraph of span n. One might ask, wether such sequences exist and how much of them exist for certain values of $n \in \mathbb{N}$? We will come back to this problem in Section 3 of this chapter. de Bruijn digraphs have a lot of applications. For example, they are used for computer network building (see for example [34]). However, in this chapter we focus on the question, wether there exists a k-regular subgraph in the de Bruijn graph of span n for a given number of vertices. We begin with an Introduction of digraphs. Then we give an overview of some basic facts of de Bruijn digraphs. In the third section of this chapter we show, that there are cycles of arbitrary length in the q-ary de Bruijn digraph. This result was obtained independently by Yoeli, Bryant, Heath, Killik, Golomb, Welch and Goldstein for the binary case ([23] and [22]). Lempel generalized this result to q-ary de Bruijn graphs in [23]. Since cycles are connected one regular digraphs, this shows that there are 1-regular subgraphs of the de Bruijn digraph of span n, for every given number of vertices. Finally the last section of this chapter is dealing with the study of k-regular subgraphs of the de Bruijn digraph, i.e. we obtain that there do not exist k-regular subgraphs for any number of vertices.

3.1 Introduction of digraphs

Some basics about digraphs

Definition of digraphs

Let $\mathcal{V}, I$ be arbitrary sets and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \times I$. We call $\Gamma := (\mathcal{V}, \mathcal{E})$ the *digraph* with vertices in $\mathcal{V}$ and edges in $\mathcal{E}$. An element $v \in \mathcal{V}$ is called a *vertex* of Γ and an element $e = (v_1, v_2, i) \in \mathcal{E}$ is called an *(directed) edge* in Γ, where e runs from the vertex v_1 to the vertex v_2. For this we write $v_1 \overset{e}{\to} v_2$ or $v_1 \overset{i}{\to} v_2$. A *loop* in Γ is an edge $v \overset{e}{\to} v$ for which the terminal vertex is equal to the initial vertex. We call an edge e in Γ, *incident to* a vertex $v \in \mathcal{V}$, if $u \overset{e}{\to} v$ for some $u \in \mathcal{V}$. We call e *incident from* v, if $v \overset{e}{\to} u$ for some $u \in \mathcal{V}$ and e is *incident at* v, if e is incident to v or incident from v. A vertex $v \in \mathcal{V}$ is called an *isolated vertex* in Γ, if there does not exist an edge which is incident at v. If $|\mathcal{V}|, |I| < \infty$, then Γ is called a *finite digraph*. If $|I| = 1$, then Γ is called a digraph without multiple edges. In this case, we suppose that $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ and we write $v_1 \to v_2$ for the edge $e = (v_1, v_2) \in \mathcal{E}$. All graphs which occurs in this survey are digraphs, therefore we use digraph, directed graph and graph simultaneously.

Subgraphs

We call a graph $\tilde{\Gamma} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ a *subgraph* of a graph $\Gamma = (\mathcal{V}, \mathcal{E})$, if $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ and $\tilde{\mathcal{E}} \subseteq \mathcal{E}$,for this we write $\tilde{\Gamma} \subseteq \Gamma$. $\tilde{\Gamma}$ is called a *spanning subgraph* of Γ, if $\tilde{\Gamma}$ has the same vertex set as Γ.

Let $\Gamma_1 := (\mathcal{V}_1, \mathcal{E}_1)$, $\Gamma_2 := (\mathcal{V}_2, \mathcal{E}_2)$ be graphs and $\Lambda := (\mathcal{V}_3, \mathcal{E}_3)$ be a subgraph of Γ_1. The union and intersection of Γ_1 with Γ_2 and the complement of Λ in Γ is defined as the graphs:

$$\begin{aligned} \Gamma_1 \cup \Gamma_2 &:= (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E}_1 \cup \mathcal{E}_2)\,, \\ \Gamma_1 \cap \Gamma_2 &:= (\mathcal{V}_1 \cap \mathcal{V}_2, \mathcal{E}_1 \cap \mathcal{E}_2)\,, \\ \Lambda^c &:= (\mathcal{V}, \mathcal{E}_1 - \mathcal{E}_2) \quad \text{whereas } \Lambda^c \subseteq \Gamma_1\,. \end{aligned}$$

Graph isomorphism

Let $\Gamma = (\mathcal{V}_1, \mathcal{E}_1), \Lambda = (\mathcal{V}_2, \mathcal{E}_2)$ be two graphs, where $\mathcal{E}_1 \subseteq \mathcal{V}_1 \times \mathcal{V}_1 \times I_1$ and $\mathcal{E}_2 \subseteq \mathcal{V}_2 \times \mathcal{V}_2 \times I_2$. We call Γ and Λ *isomorph* graphs and write $\Gamma \cong \Lambda$, if there exists a bijective map $\phi : \mathcal{V}_1 \leftrightarrow \mathcal{V}_2$ such that

$$|\{i \in I_1 | (v_1, v_2, i) \in \mathcal{E}_1\}| = |\{j \in I_2 | (\, \phi(v_1), \phi(v_2), j\,) \in \mathcal{E}_2\}| \quad \text{for all } \; v_!, v_2 \in \mathcal{V}_1 \, .$$

This means for graphs without multiple edges, that $(v_1, v_2) \in \mathcal{E}_1$ holds if and only if $(\, \phi(v_1), \phi(v_2)\,) \in \mathcal{E}_2$ hold. The map ϕ is called a *graph isomorphism.*

Vertex degree

Let $\Gamma := (\mathcal{V}, \mathcal{E})$ be a finite graph and v be a vertex in Γ. We denote with $d_i(v)$ the *indegree* and with $d_o(v)$ the *outdegree* of the vertex v. This is the total number of edges which are incident to v respectively incident from v. We write the vertex v in Γ has *degree* $d(v)$, if the total number of edges which are incident at v is d and for this we write $d(v) = d$. It follows that

$$d(v) = d_i(v) + d_o(v) - \textit{numbers of loops at } v \, .$$

We call Γ a *q-regular* digraph, if $d_i(v) = d_o(v) = q$ for all $v \in \Gamma$. This means that for every vertex v there exists exactly q edges of Γ with initial vertex v. and q edges of Γ with terminal vertex v. If Γ is a q-regular graph with L vertices, Γ contains qL edges. Vice versa, if Γ is q-regular with M edges, Γ contains $\frac{M}{q}$ vertices.

A digraph $\Gamma := (\mathcal{V}, \mathcal{E})$ is called an *Euler graph*, if for every vertex $v \in \mathcal{V}$ we have $d_i(v) = d_o(v)$. Obviously every regular graph is also an Euler graph.

Walks in a graph

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a digraph. A *walk* P in Γ of length n, from v_1 to v_{n+1}, is a sequence $v_1 e_1 v_2 \dots v_n e_n v_{n+1}$ with $v_1, \dots, v_{n+1} \in \mathcal{V}$ and $v_i \xrightarrow{e_i} v_{i+1}$ is an edge of Γ for all $1 \leq i \leq n$. For this we write:

$$v_1 \xrightarrow{P} v_{n+1} \;\text{ or }\; v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \dots \xrightarrow{e_n} v_{n+1} \, .$$

Particulary a length 0 walk is a single vertex of Γ. We denote with $|P| = n$ the length of P. If $|P| > 0$, the walk P is uniquely defined by its sequence of edges $e_1 \dots e_n$.If moreover Γ is a graph without multiple edges, P is also uniquely determined by the sequence of vertices $v_1 \dots v_{n+1}$. In this case we cease sometimes

the edges over the arrows in the notation above.

The walk P is called a *closed walk*, if $v_1 = v_{n+1}$. P is called a *path*, if P runs through every edge one time at the most. This means $e_i \neq e_j$ for all $i \neq j$. P is called a *cycle*, if it is a closed walk and runs trough every vertex different to the start vertex not more than one time. This means $v_1 = v_{n+1}$ and $v_i \neq v_j$ for all $1 \leq i < j \leq n$. Furthermore P is called a *simple path* if every vertex in P occurs exactly one time or if P is a cycle. This means that P is a simple path if $v_i \neq v_j$ for all $i \neq j$ or if P is a cycle. Obviously any simple path is also a path.

The associated graph $\mathcal{P} = (\mathcal{V}_P, \mathcal{E}_p)$ of the walk P is given by:

$$\mathcal{V}_P := \{v_1, \ldots, v_{n+1}\} \text{ and } \mathcal{E}_P := \{e_1, \ldots, e_n\}.$$

We obtain the following relations between a walk P and its associated subgraph:

$$\begin{array}{lcl} P \text{ is a closed path} & \Leftrightarrow & \mathcal{P} \text{ is a connected Euler graph,} \\ P \text{ is a cycle} & \Leftrightarrow & \mathcal{P} \text{ is a connected 1-regular graph.} \end{array}$$

If P is a simple path, but not a cycle, then every vertex in $\mathcal{P}$ expect v_{n+1} has a unique successor vertex in $\mathcal{P}$ and every vertex in $\mathcal{P}$ expect v_1 has a unique antecessor vertex in $\mathcal{P}$. Therefore a simple path which is not a cycle is uniquely determined by its associated graph. Furthermore if the starting respectively the end vertex in a cycle is not important, we can interpret a cycle also as its associated one-regular subgraph. Therefore, if the context is clear, we don't distinguish between a simple pathes its associated subgraphs.

We call a closed path E in Γ an *Euler circuit* of Γ, if E runs through every edge of Γ exactly one time. A cycle C is called a *Hamilton circuit* if C runs through every vertex of Γ exactly one time.

Let $P_1 = v_1 e_1 \ldots e_n v_{n+1}$ be a walk of length n and $P_2 = \tilde{v}_1 \tilde{e}_1 \ldots \tilde{e}_n \tilde{v}_{m+1}$ be a walk of length m. If the end vertex of P_1 is equal to the starting vertex of P_2, we define $P_1 P_2$ as the length $(n+m)$ walk given by the concatenation of the two pathes:

$$P_1 P_2 := v_1 e_1 \ldots e_n \tilde{v}_1 \tilde{e}_1 \ldots \tilde{e}_n \tilde{v}_{m+1}.$$

Factors of a graph

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph and $\Lambda = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ be a finite subgraph of Γ. The subgraph Λ is called a *q-factor* of Γ, if Λ is a q-regular graph and a spanning subgraph in Γ. This means $\tilde{\mathcal{V}} = \mathcal{V}$ and $d_o(v) = d_i(v) = q$ for all vertices v in Λ. The next proposition is obviously.

Proposition 23 *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph without multiple edges and $\Lambda = (\mathcal{V}, \tilde{\mathcal{E}})$ be a finite spanning subgraph of Γ. The following conditions are equivalent.*

(i) Γ *is a* 1*-factor in* Γ*.*

(ii) Λ *is the union of vertex disjoint cycles.*

(iii) There exists a bijective map $\phi : \mathcal{V} \leftrightarrow \mathcal{V}$, such that

$$(v, u) \in \tilde{\mathcal{E}} \Leftrightarrow u = \phi(v)$$

holds for all $u, v \in \mathcal{V}$.

Furthermore every q-regular graph is the union of vertex disjoint cycles.

Connected graphs, connected components and Euler graphs

Let $\Gamma := (\mathcal{V}, \mathcal{E})$ be a graph. A vertex $v \in \mathcal{V}$ is called *reachable* from another vertex $u \in \mathcal{V}$, if there exists a walk from u to v. This is the same as to say that there exists a simple path from u to v. The *distance* between u and v in Γ is given by

$$dist(u, v) := \min\{|P| \mid u \xrightarrow{P} v \ P \text{ is simple path }\},$$

if v is reachable from u and $dist(u, v) := \infty$ otherwise. Let

$$\Gamma^n(v) := \{u \in \mathcal{V} | \text{there is a simple path of length } n \text{ from } v \text{ to } u\}$$

be the set of all vertices which have distance n from $v \in \mathcal{V}$ in Γ. Furthermore we define

$${}^n\Gamma(v) := \{u \in \mathcal{V} | \text{there is a simple path of length } n \text{ from } u \text{ to } v\,.\}$$

Then ${}^n\Gamma(v)$ is the set of all vertices, from which the distance to v is n. We call the vertices in $\Gamma^1(v)$ the *successors* of v in Γ and the vertices in ${}^1\Gamma(v)$ the *antecessors* of v. Let $A \subseteq \mathcal{V}$ be an arbitrary set of vertices. Then we define the successor set of A as $\Gamma^n(A) := \bigcup_{v \in A} \Gamma^n(v)$ and the antecessor set of A as ${}^n\Gamma(A) = \bigcup_{v \in A} {}^n\Gamma(v)$.

Instead of $\Gamma^1(v)$ and $\Gamma^1(A)$ we write commonly $\Gamma(v)$ and $\Gamma(A)$ respectively. By this we have

$$|\Gamma(v)| = d_o(v) \quad \text{and} \quad \left|{}^1\Gamma(v)\right| = d_i(v)$$

It follows that Γ is q-regular if and only if $|\Gamma(v)| = |{}^1\Gamma(v)| = q$ for all $v \in \mathcal{V}$.

We call Γ *(strongly) connected*, if u is reachable from v for all $u, v \in \mathcal{V}$. In the most books about graph theory, a digraph is called connected, if the underlying undirected graph is connected and a digraph is called strongly connected if it fulfill the condition above. Since we pay no attention to undirected graphs in this survey, we cease the "strongly". This means that in this survey a connected graph Γ is a digraph for which there exists a simple path between every two vertices of Γ.

Let $\Gamma := (\mathcal{V}, \mathcal{E})$ be a graph. Γ can be *split into cycles* if there exists a set $\mathcal{C}$ of edge disjoint cycles, such that Γ is the edge disjoint union of the cycles in $\mathcal{C}$. This means two different cycles in $\mathcal{C}$ have no common edge and the cycles in $\mathcal{C}$ covers Γ, where we understand an isolated vertex as a cycle of length 0. We call the set $\mathcal{C}$ a *cycle splitting* of Γ.

Proposition 24 *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a finite graph.Then Γ is an Euler graph if and only if Γ can be split into cycles.*

Proof: Let Γ be a finite Euler graph. We show that there exists at least one cycle in Γ. We choose an arbitrary vertex $v_1 \in \mathcal{V}$, if v_1 is isolated, we have found a cycle of length 0. If v_1 is not isolated, than there exists at least one edge which is incident at v_1. Since $d_i(v_1) = d_o(v_1) \geq 1$, there exists also an edge e_1 with initial vertex v_1. Let v_2 be the terminal vertex of e_1, then $v_1e_1v_2$ is a simple path in Γ. If $v_1 = v_2$ we have found a cycle. Let $v_1e_1v_2 \dots e_nv_{n+1}$ be a simple path of length n, which is not a cycle. We have $d_i(v_{n+1}) \geq 1$, because Γ is an Euler graph there exists an edge e_{n+1} incident from v_{n+1}. Let v_{n+2} the terminal vertex of e_{n+1}. If $v_{n+2} \neq v_i$ for all $1 \leq i \leq (n+1)$ we obtain that $v_1e_1v_2 \dots v_{n+1}e_{n+1}v_{n+2}$ is a simple path of length $(n+1)$ which is not a cycle. If we continue with this procedure, it follows that at some point $v_{n+2} = v_i$ for some $1 \leq i \leq n+1$, because Γ has only a finite number of vertices. Since $v_1e_1 \dots e_nv_{n+1}$ is a simple path,we obtain that $v_ie_i \dots e_nv_{n+1}e_{n+1}v_{n+2}$ is a cycle. Therefore every finite Euler graph obtain at least one cycle. If we delete in Γ the edges of the cycle, we obtain a new graph Γ which is also an Euler graph. While Γ is finite, we obtain by induction a finite numbers of edge disjoint cycles which covers Γ. This shows that there exists a cycle splitting for Γ.

Thus let Γ be a finite graph and $\mathcal{C}$ be a cycle splitting of Γ. Let v be a none isolated vertex of Γ. Then there exists cycles $C_1, \ldots, C_m \in \mathcal{C}$, such that every edge incident at v is contained in one of the cycles. While a cycle is 1-regular and the cycles in $\mathcal{C}$ are edge disjoint, it follows that $d_i(v) = d_o(v) = m$. This shows that Γ is an Euler graph. **q.e.d**

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a finite Euler graph and $u, v \in \mathcal{V}$. If u is reachable from v, then there exists a simple walk

$$v_1 e_1 v_2 \ldots v_n e_n v_{n+1} \text{ with } v_1 = v \text{ and } v_{n+1} = u\,.$$

Let $\mathcal{C} := \{C_1, \ldots, C_k\}$ be a cycle splitting of Γ. Then there exists for every $i \in \{1, \ldots, n\}$ a (unique) cycle $C \in \mathcal{C}$, such that e_i is an edge in C. Let us denote with with P_i the simple path, which is obtained by deleting e_i in C. It follows that $P_n P_{n-1} \ldots P_1$ is a walk from u to v. Therefore v is also reachable from u. This shows:

Proposition 25 *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a finite Euler graph. Then we have for every $u, v \in \mathcal{V}$:*

$$u \textit{ is reachable from } v \Leftrightarrow v \textit{ is reachable from } u\,.$$

If u is reachable from v and w is reachable from u then obviously w is reachable from v. This shows that the relation "reachable" is an equivalent relation on the vertex set of a finite Euler graphs. Let $\{\mathcal{V}_1, \ldots, \mathcal{V}_m\}$ be the partition of the vertex set $\mathcal{V}$ of Γ given by this equivalent relation. i.e. the $\mathcal{V}_i$s are the equivalent classes. Let $\mathcal{E}_i$ be the edge sets given by

$$(u, v) \in \mathcal{E}_i \text{ iff } u, v \in \mathcal{V}_i \quad i \in \{1, \ldots, m\},.$$

Then $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ is a partition of the edge set $\mathcal{E}$ of Γ. Furthermore it follows that $\Gamma_i := (\mathcal{V}_i, \mathcal{E}_i)$ is a connected Euler subgraph of Γ for all $1 \le i \le m$ and that there does not exists connections in Γ between two different of this subgraphs.

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be an arbitrary graph. We call a collection $\Gamma_1, \ldots \Gamma_m \subseteq \Gamma$ of subgraphs a *decomposition into the connectivity components* of Γ, if the following properties hold:

(1) $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_m$,

(2) all Γ_i's are connected graphs,

(3) there does not exist connections in Γ between the subgraphs $\Gamma_1, \ldots, \Gamma_m$. Especially for every vertex u in Γ_i and vertex v in Γ_j u isn't reachable from v in Γ and vice versa.

Obviously such a decomposition is uniquely. We have shown above, that every finite Euler graph has a decomposition into connected components. Furthermore every connected component of an Euler graph is by itself an Euler graph.

For finite Euler graphs the well known theorem from Euler holds:

Theorem 8 (Euler) *Let Γ be a finite graph without isolated vertices. Then*

$$\Gamma \textit{ has an Euler circuit } \Leftrightarrow \Gamma \textit{ is a connected Euler graph },$$

i.e. for every regular connected finite graph exists an Euler circuit.

Proof: Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a finite graph without isolated vertices and E be an Euler circuit for Γ. Let $u, v \in \mathcal{V}$. Since u, v are none-isolated vertices and E runs through every edge of Γ one time it follows that u and v occurs at least once in E. Let us suppose that u occurs before v in the sequence E, then v is reachable from u. While E is a closed walk we obtain, that u is also reachable from v. Therefore Γ is a connected graph. If v occurs in E m-times, then there exist exactly m edges in Γ with initial vertex v and m edges with terminal vertex v, because E is a closed walk which runs through every edge of Γ exactly one time. It follows that Γ is an Euler graph.

Thus let Γ be a finite connected Euler graph without isolated vertices. We proof by induction on the number of edges of Γ, that there exists an Euler circuit in Γ. If Γ has only one edge, then Γ consists of a single vertex and a loop at this vertex. In this case, the loop is an Euler circuit for Γ. Let us assume, that for $n > 1$, every finite connected Euler graph without isolated vertices and k edges has an Euler circuit, if $k < n$. Let Γ be a finite connected Euler graph with n edges and without isolated vertices. By Proposition 24 follows, that there exists a cycle C (with length bigger than 0) in Γ. If we delete the edges of this cycle in Γ, we obtain a new graph $\tilde{\Gamma}$ which is also an Euler graph, but $\tilde{\Gamma}$ has less than k edges. In general, this graph is not a connected graph , but every connectivity component of $\tilde{\Gamma}$ is a connected Euler graph with less than k edges. By the induction hypothesis follows, that every connectivity component of $\tilde{\Gamma}$ has an Euler circuit. While Γ is a connected graph, every connectivity component of $\tilde{\Gamma}$ has at least one vertex on the cycle C. We obtain an Euler circuit for Γ, by travelling along the cycle. If we come to a vertex v in a connectivity component of $\tilde{\Gamma}$ which we haven't visited before, we stop and run through the corresponding Euler circuit of the connectivity component. After one round we come back to v and continue to travel around the cycle C. If we finish one round in C, we had visited every edge in Γ exactly one time. This gives us an Euler circuit of Γ. **q.e.d**

Factors of q-regular digraphs

Let us recall Hall's matching theorem. A graph $\Lambda = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ is called a *bipartite digraph*, if there exists a partition $\mathcal{V}_1, \mathcal{V}_2$ of $\mathcal{V}$, such that all edges in Λ have their initial vertex in $\mathcal{V}_1$ and their terminal vertex in $\mathcal{V}_2$. Furthermore we allow multiple edges. A *matching set* in Λ is an edge set $M \subseteq \mathcal{E}$, such that any two different edges in M have neither the same initial vertex nor the same terminal vertex. A *complete matching* in Λ is a matching set M, such that for every $v \in \mathcal{V}_1$ there exists an edge in M with initial vertex v.

Theorem 9 (Hall's matching theorem) *Let $\Lambda = (\mathcal{V}_1 \cup \mathcal{V}_2, \tilde{\mathcal{E}})$ be a finite bipartite digraph. There exists a complete matching set $M \subseteq \mathcal{E}$ for Γ if and only if $|\Gamma(A)| \geq |A|$ for all $A \subseteq \mathcal{V}_1$*

A proof of the theorem above should be found in nearly every book about graph theory or combinatorics, for example [28].

Corollary 2 *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a finite q-regular digraph. Then there exists a 1-factor in Γ.*

Proof: Let $\Gamma := (\mathcal{V}, \mathcal{E})$ be a finite q-regular graph, with $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \times I$, $|I| < \infty$. We define a bipartite digraph $\Lambda = (\mathcal{V}_1 \cup \mathcal{V}_2, \tilde{\mathcal{E}})$ as follows. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two different duplicates of $\mathcal{V}$. This means $\mathcal{V}_1 := \mathcal{V} \times \{1\}$ and $\mathcal{V}_2 := \mathcal{V} \times \{2\}$. We define the edge set $\tilde{\mathcal{E}} \subseteq \mathcal{V}_1 \times \mathcal{V}_2 \times I$ as follows.

> For $u, v \in \mathcal{V}, i \in I$ the edge $(v, 1) \xrightarrow{i} (u, 2)$ exists in Λ if and only if $v \xrightarrow{i} u$ is an edge in Γ.

Obviously Λ is a finite bipartite digraph. Since Γ is q-regular it follows that

$$d_o(v) = q = d_i(u) \text{ for all } v \in \mathcal{V}_1, u \in \mathcal{V}_2 \,.$$

Let $A \subseteq \mathcal{V}_1$. It follows that there are $q|A|$ edges with initial vertex in A and terminal vertex in $\Lambda(A)$, but that there exists totally $q|\Lambda(A)|$ edges with terminal vertex in $\Lambda(A)$. Therefore we obtain $|A| \leq |\Lambda(A)|$ for all $A \subseteq \mathcal{V}_1$. ¿From Hall's matching theorem follows, that there exists a complete matching set $M \subseteq \mathcal{V}_1$ for Λ. This means, for every $v \in \mathcal{V}$, there exists unique $u \in \mathcal{V}, i \in I$, such that $(v, 1) \xrightarrow{i} (u, 2)$ is an edge in M. With $|\mathcal{V}_1| = |\mathcal{V}| = |\mathcal{V}_2|$ we conclude, that also for every $u \in \mathcal{V}$, there are unique $v \in \mathcal{V}, i \in I$, such that $(v, 1) \xrightarrow{i} (u, 2)$ is an edge in M. Thus we obtain a 1-factor of Γ, if we identify the edges in M with its corresponding edges in $\mathcal{E}$. **q.e.d**

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a q-regular digraph and $\Lambda = (\mathcal{V}, \mathcal{E}')$ be a k-factor of Γ. Obviously the subgraph $\Lambda^c = (\mathcal{V}, \mathcal{E} - \mathcal{E}') \subseteq \Gamma$ is a $(q-k)$-factor of Γ, i.e. Λ^c is $(q-k)$-regular. Vice versa, let Λ_1 be a k_1-factor of a graph Γ and Λ_2 be a k_2-factor of Γ. If Λ_1 and Λ_2 are edge disjoint, then $\Lambda_1 \cup \Lambda_2$ is a $(k_1 + k_2)$-factor of Γ. Therefore we obtain the following proposition with the help of Corollary 2.

Proposition 26 *Let $\Gamma := (\mathcal{V}, \mathcal{E})$ be a finite q-regular graph.*

(i) There exists a k-factor of Γ, for every $1 \leq k \leq q$.

(ii) Let Λ be a k-factor of Γ and $1 \leq m \leq k$. A subgraph $\tilde{\Lambda} \subseteq \Lambda$ is a m-factor of Λ if and only if $\tilde{\Lambda}$ is a m-factor of Γ.

(iii) Let $k_1, \ldots, k_m \in \mathbb{N}$, such that $k_1 + \ldots + k_m \leq q$. Then there exists edge disjoints factors $\Lambda_1, \ldots, \Lambda_k$ of Γ, with Λ_i is a k_i-factor for all $1 \leq i \leq m$. If $k_1 + \ldots + k_m = q$, then Γ is the edge disjoint union of the Λ_i's. Especially there exists an edge disjoint decomposition of Γ into q 1-factors.

(iv) If Λ is a m-factor of Γ and $1 \leq k \leq m$, then there exists a k-factor $\tilde{\Lambda}$ of Γ with $\tilde{\Lambda} \subseteq \Lambda$.

(v) If Λ is a k-factor of Γ and $1 \leq k \leq m \leq q$, then there exists a m-factor $\tilde{\Lambda}$ of Γ with $\Lambda \subseteq \tilde{\Lambda}$.

Moreover part (ii) and (iv) in the proposition above holds for any digraph Γ.

Linegraphs

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph. The *linegraph* $L\Gamma := (\mathcal{V}_1, \mathcal{E}_1)$ is defined as:

(i): $\mathcal{V}_1 := \mathcal{E}$. This means, the vertices of $L\Gamma$ are the edges in Γ.

(ii): Let $e_1, e_2 \in \mathcal{V}_1$ be two vertices of $L\Gamma$. There exists an edge from e_1 to e_2 in $L\Gamma$ if and only if $e_1 e_2$ is a walk in Γ (of length 2).

$$\mathcal{E}_1 := \{(e_1, e_2) \in \mathcal{E}^2 | \text{ the terminal vertex of } e_1 \text{ is the initial vertex of } e_2\}$$

This means, that the edges in $L\Gamma$ are the walks of length 2 in Γ. We observe that $L\Gamma$ has no multiple edges.

We define the *k-iterated linegraph* of Γ recursively as:

$$L_{k+1}\Gamma = L(L_k\Gamma) \text{ with } L_0\Gamma := \Gamma\,.$$

By an easy induction we obtain:

$$L_{k+m}\Gamma = L_k(L_m\Gamma) = L_m(L_k\Gamma) \quad \text{for } k, m \in \mathbb{N}_0\,.$$

By induction it is easy to verify, that the vertices of $L_k\Gamma$ can be labelled with the walks in Γ of length k and that the edges of $L_k\Gamma$ can be labelled with the walks in Γ of length $(k+1)$, in the following way:

> Let u, u' be two vertices in $L_k\Gamma$, where u should be labelled with the walk $P = v_1e_1v_2 \dots v_ne_nv_{n+1}$ in Γ and u' should be labelled with the walk $P' = v'_1e'_1v'_2 \dots v'_ne'_nv'_{n+1}$ in Γ. Then there exists an edge from u to u' in $L_k\Gamma$ if and only if $v_2e_2v_3 \dots v_ne_nv_{n+1} = v'_1e'_1v'_2 \dots v'_{n-1}e'_{n-1}v'_n$. Furthermore this edge is labelled with the walk of length $(k+1)$ given by:
>
> $$v_1e_1v_2 \dots v_{n+1}e_nv'ne'_nv'_{n+1} = v_1e_1v'_1e'_1v'_2 \dots v'_ne'_nv'_{n+1}\,.$$

More precisely:

Let us understand walks as sequences of edges. We define:

$$\begin{aligned}
\mathcal{V}_k &:= \{P | P = e_1 \dots e_k \text{ walk of length k in } \Gamma\}\,,\\
\mathcal{E}_k &:= \{(e_1 \dots e_k, e_2 \dots e_{k+1}) \subseteq \mathcal{E}^k \times \mathcal{E}^k |\, e_1 \dots e_{k+1} \text{is a length } (k+1) \text{ walk in } \Gamma\}\,.
\end{aligned}$$

Then an easy induction proof shows, that $L_k\Gamma \cong (\mathcal{V}_k, \mathcal{E}_k)$ and that

$$(e_1, \dots e_k, e_2 \dots e_{k+1}) \in \mathcal{E}_k \leftrightarrow e_1 \dots e_{k+1} \text{ length k+1 walk in } \Gamma$$

gives us a one-to-one relation between the walks of length $(k+1)$ in Γ and the edges of $L_k\Gamma$.

If Λ is a subgraph of Γ, then obviously $L_k\Lambda$ is a subgraph of $L_k\Gamma$. Furthermore for Linegraphs the following proposition holds. Since the proposition is mostly obviously, we omit a proof.

Proposition 27 *Let $\Gamma := (\mathcal{V}, \mathcal{E})$ be a graph and $L_k\Gamma = (\mathcal{V}_k, \mathcal{E}_k)$ be the k-iterated linegraph of Γ.*

(i) *$L_k\Gamma$ has no multiple edges and no isolated vertices for all $k \geq 1$.*

(ii) *$e \in \mathcal{E}$ is a loop in Γ if and only if there exists a loop at the vertex e in $L\Gamma$, i.e. the number of loops in $L_k\Gamma$ is equal to the number of loops in Γ.*

(iii) *If Γ is q-regular then also $L_k\Gamma$ is q-regular, as well. Moreover, if Γ is finite, then $|\mathcal{V}_k| = q^k|\mathcal{V}_k|$*

(iv) *Γ is a connected graph if and only if $Ł_k\Gamma$ is a connected graph for some $k \in \mathbb{N}_0$, i.e. $L_k\Gamma$ is a connected graph for all $k \in \mathbb{N}_0$, if Γ is a connected graph.*

(v) *If Λ is a q-regular subgraph of Γ with p vertices, then $Ł_k\Lambda$ is a q-regular subgraph of $L_k\Gamma$ with q^kp vertices.*

(vi) *If Λ_1 ,Λ_2 are edge disjoint subgraphs of Γ, then $L_k\Lambda_1$ and $L_k\Lambda_2$ are vertex disjoint subgraphs of $L_k\Gamma$ for all $k \geq 1$.*

(vii) *Let $\mathcal{C}$ be a cycle of length k in Γ, then $L_k\mathcal{C}$ is also a cycle of length k in $L_k\Gamma$.*

(viii) *Let Γ be q-regular and $\Lambda_1, \dots, \Lambda_q$ be edge disjoint 1-factors of Γ. Then $L\Lambda_1, \dots, L\Lambda_q$ are vertex disjoint and $(L\Lambda_1 \cup \ldots \cup L\Lambda_q)$ is a 1-factor of $L\Gamma$.*

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph and $P = e_1 \dots e_n$ be a walk of length n in Γ with $e_i \in \mathcal{E}\ \forall 1 \leq i \leq n$. We denote with LP the *linewalk* walk of length $(n-1)$ in the linegraph $L\Gamma$ which is given by:

$$LP := (e_1, e_2) \dots (e_{n-1}, e_n)\,, \qquad \text{whereas the } (e_i, e_{i+1}) \text{ are all edges in } L\Gamma.$$

Obviously P is a walk of length $(n-1)$ in $L\Gamma$. Furthermore we define recursively L_kP as:

$$L_{k+1}P := L(L_kP)\,, \quad L_0P := P \quad \text{for all } 1 \leq k \leq n\,.$$

Obviously L_kP is a walk of length $(n-k)$ in $L\Gamma$ for all $1 \leq k \leq n$. Especially L_nP is a single vertex in $Ł_n\Gamma$.

If P is a closed walk, then $e_n e_1$ is a walk of length 2 in Γ. Therefore we can define for closed walks, the *closed linewalk* $\hat{L}P$ as:

$$\hat{L}P := (e_1, e_2) \dots (e_{n-1}, e_n)(e_n, e_1) \, .$$

Obviously $\hat{L}P$ is a closed walk in ŁΓ of length n and $\hat{L}P = (LP)(e_n, e_1)$. We define recursively:

$$\hat{L}_{k+1}P := L(\hat{L}_k P) \, , \quad \hat{L}_0 P := P$$

It follows that $\hat{L}_k P$ is a closed walk in $L_k\Gamma$ of length n.

We obtain the following proposition, which is obviously for the most part.

Proposition 28 *Let $\Gamma : (\mathcal{V}, \mathcal{E})$ be a graph.*

(i) If P_1, P_2 are two edge disjoints walks in Γ, then LP_1 and LP_2 are vertex disjoints walks in $L\Gamma$. The same holds for closed walks and $\hat{L}$.

(ii) If P is a path in Γ, then LP is a simple path in LP

(iii) If P is a closed path (of length n) in Γ, then $\hat{L}_k P$ is a cycle (of length n) for all $k \geq 1$

(iv) If P is a cycle in Γ with corresponding subgraph $\mathcal{P}$, then $L_k\mathcal{P}$ is the the corresponding subgraph of $\hat{L}_k P$ in $L_k\Gamma$.

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph and Λ be a finite Euler subgraph in Γ with n edges. There exists an Euler circuit E for Λ. Especially E is a closed path of length n in Γ. From part (iii) of the proposition above follows that $\hat{L}_k P$ is a cycle of length n in $L_k\Lambda \subseteq L_k\Gamma$ for all $k \in \mathbb{N}$. Especially $\hat{L}_k P$ is a Hamilton circuit of $L_k\Lambda$.

Let P' be a cycle of length n in $L\Gamma$. Since $L\Gamma$ has no multiple edges, we can understand P' as a sequences of vertices in $L\Gamma$:

$$P = e_1 \dots e_n e_1 \text{ with } e_1, \dots, e_n \text{ vertices in } L\Gamma \, .$$

While the vertices in $L\Gamma$ are the edges of Γ, we can understand the sequence $e_1 \dots e_n$ of edges in Γ as a walk of length n in Γ. Furthermore it follows that $e_1 \dots e_n$ is a closed path in Γ, because P is a cycle in $L\Gamma$. If we denote with P' the closed path $e_1 \dots e_n$ in Γ, then follows that $\hat{L}P' = P$. Furthermore we define $\hat{L}_* P$ as the subgraph in Γ which corresponds to the closed path P'. Since P is a closed path of length n, we obtain that $\hat{L}_* P$ is an Euler subgraph in Γ with n edges. Let E be an Euler circuit for an Euler subgraph $\Lambda \subseteq \Gamma$ with n edges. We have already shown above, that $\hat{L}E$ is a cycle in $L\Gamma$ of length n. Therefore $\hat{L}_* \hat{L}E$

is defined. Furthermore it is easy to verify, that $\hat{L}_* \hat{L} E = \Gamma$. Therefore $\hat{L}_*$ maps the cycles of length n in $L\Gamma$ onto the Euler subgraphs in Γ with n vertices. Let Λ be a Euler subgraph in Γ with n vertices and edge set $\mathcal{E}_\Lambda$. It follows that there exists a cycle of length n in $L\Gamma$ if and only if there exists an Euler subgraph in Γ with n vertices. Furthermore we have:

$$\hat{L}_*^{-1}\Lambda = \{P = e_1 \dots e_n e_1 \,|\, e_1 \dots e_n \in \mathcal{E}_\Lambda^n \text{ is a closed path } \} \tag{3.1}$$

3.2 The de Bruijn digraph $\mathcal{B}_{\mathcal{A}}(n)$

Definition of de Bruijn digraphs

Let $\mathcal{A}$ be an arbitrary alphabet, where we allow infinite alphabets. We define for $n \in \mathbb{N}$ the *n-th level de Bruijn Graph* $\mathcal{B}_{\mathcal{A}}(n) = (\mathcal{V}, \mathcal{E})$ as follows:

1. The vertices of $\mathcal{B}_{\mathcal{A}}(n)$ are the words over $\mathcal{A}$ of length n. This means $\mathcal{V} := \mathcal{A}^n$.

2. Let $w, w' \in \mathcal{A}^n$. There is an edge from w to w' in $\mathcal{B}_{\mathcal{A}}(n)$ if and only if the letters of w at the $(i+1)$-th position is equal to the letter of w' at the i-th position for all $1 \leq i \leq n-1$. Therefore the edge set of $\mathcal{B}_{\mathcal{A}}(n)$ is given by:

$$\mathcal{E} := \{(au, ub) \in \mathcal{A}^n \times \mathcal{A}^n | a, b \in \mathcal{A}, u \in \mathcal{A}^{n-1}\} = \bigcup_{w \in \mathcal{A}^n} \left(w, \mathcal{A}^{-1} w \mathcal{A}\right) .$$

   This means $w_1 \dots w_n \to w_2 \dots w_{n+1}$ is an edge in $\mathcal{B}_{\mathcal{A}}(n) = (\mathcal{V}, \mathcal{E})$ for all $w_1, \dots, w_{n+1} \in \mathcal{A}$.

Obviously $\mathcal{B}_{\mathcal{A}}(n)$ has no multiple edges for $n \in \mathbb{N}$. If n=0, the graph $\mathcal{B}_{\mathcal{A}}(0)$ is defined as the multiple edge digraph which has the empty sequence $e \in \mathcal{A}^0$ as its only vertex and for every $a \in \mathcal{A}$ there exists a loop $e \xrightarrow{a} e$ in $\mathcal{B}_{\mathcal{A}}(0)$.

If $\mathcal{A} = \{0, 1, \dots, q\}$, then we will write $\mathcal{B}_q(n)$ in place of $\mathcal{B}_{\mathcal{A}}(n)$. If $|\mathcal{A}| = q$ for some arbitrary finite alphabet, then we can understand $\mathcal{B}_{\mathcal{A}}(n)$ also as the graph $\mathcal{B}_q(n)$, because a bijective map between $\mathcal{A}$ and $\{0, 1, \dots, q\}$ gives us a isomorphism between $\mathcal{B}_{\mathcal{A}}(n)$ and $\mathcal{B}_q(n)$. Especially we obtain $\mathcal{B}_{\mathcal{A}}(n) \cong \mathcal{B}_q(n)$.

Let $u \in \mathcal{A}^{n-1}$, $a, b \in \mathcal{A}$ for some arbitrary finite or infinite alphabet $\mathcal{A}$. We will write for the edge from au to ub in $\mathcal{B}_{\mathcal{A}}(n)$ sometimes $au \xrightarrow{b} ub$. Furthermore we obtain that $u\mathcal{A}$ as the set of successors of the vertex au and $\mathcal{A}u$ is the set of antecessors of the vertex ub. If $\mathcal{X} \subseteq \mathcal{A}^n$ is a set of vertices of $\mathcal{B}_{\mathcal{A}}(n)$, then it follows that

$$\begin{array}{ll} \mathcal{A}^{-1}\mathcal{X}\mathcal{A} & \text{is the set of successor of vertices in } \mathcal{X}, \\ \mathcal{A}\mathcal{X}\mathcal{A}^{-1} & \text{is the set of antecessors of vertices in } \mathcal{X}. \end{array}$$

Let $|\mathcal{A}| = q < \infty$. Obviously $\mathcal{B}_{\mathcal{A}}(0)$ is a q-regular graph. Let $n \in \mathbb{N}$ and $w \in \mathcal{A}^n$. It follows that

$$d_i(w) = |\mathcal{A}w\mathcal{A}^{-1}| = |\mathcal{A}| = q = |\mathcal{A}| = |\mathcal{A}^{-1}w\mathcal{A}| = d_o(w) .$$

Therefore $\mathcal{B}_{\mathcal{A}}(n)$ is a q-regular graph for all $n \in \mathbb{N}$. Especially $\mathcal{B}_q(n)$ is a q-regular graph. Since $\mathcal{A}^n$ is the vertex set of $\mathcal{B}_{\mathcal{A}}(n)$, it follows that $\mathcal{B}_{\mathcal{A}}(n)$ has q^{n+1} edges.

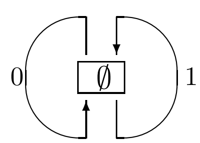
Let $\mathcal{A}$ be an arbitrary alphabet and $w, v \in \mathcal{A}^n$, where $w = w_1 \dots w_n$, $v = v_1 \dots v_n$ with $w_i, v_i \in \mathcal{A}$ for all i. We obtain a walk of length n from w to v by

$$w \xrightarrow{v_1} w_2 \dots w_n v_1 \xrightarrow{v_2} w_3 \dots w_n v_1 v_2 \xrightarrow{v_3} \dots \xrightarrow{v_{n-1}} w_n v_1 \dots v_{n-1} \xrightarrow{v_n} v\,.$$

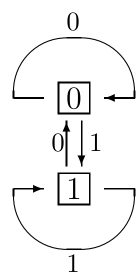
This shows that for any two vertices w, v in $\mathcal{B}_{\mathcal{A}}(n)$ there is a walk of length n from w to v. Especially this shows that $\mathcal{B}_{\mathcal{A}}(n)$ is a connected graph.

The pictures below show the graphs $\mathcal{B}_2(0)$ - $\mathcal{B}_2(3)$ and $\mathcal{B}_3(2)$.

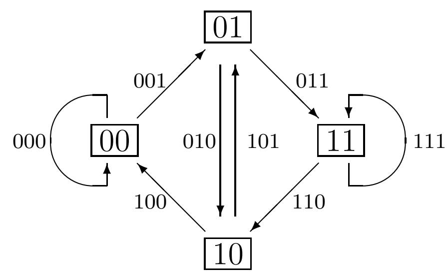
$\mathcal{B}_2(0)$



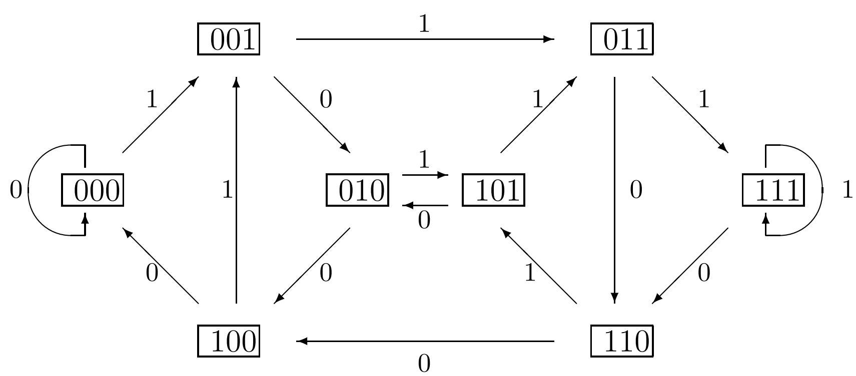
$\mathcal{B}_2(1)$



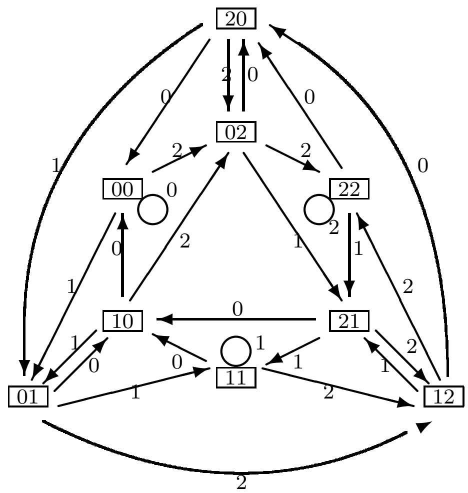
$\mathcal{B}_2(3)$



$\mathcal{B}_2(3)$



$\mathcal{B}_3(2)$



In the picture of $\mathcal{B}_2(3)$ the edges are labelled with the words of length 3. In general it is possible to label the the edges in $\mathcal{B}_{\mathcal{A}}(n)$ with the words in $\mathcal{A}^{n+1}$. If $n = 0$ we can choose an arbitrary bijection between the edges of $\mathcal{B}_{\mathcal{A}}(0)$ and $\mathcal{A}$. Thus let $n \in \mathbb{N}$ and $\mathcal{B}_{\mathcal{A}}(n) = (\mathcal{A}^n, \mathcal{E})$. Let $w_1, \ldots, w_{n+1} \in \mathcal{A}$, we obtain a one-to-one relation between the edges in $\mathcal{B}_{\mathcal{A}}(n)$ and the set $\mathcal{A}^{n+1}$ of words of length $(n+1)$ by:

$$w_1 \ldots w_{n+1} \in \mathcal{A}^{n+1} \longleftrightarrow (w_1 \ldots w_n \overset{w_{n+1}}{\to} w_2 \ldots w_{n+1}) \in \mathcal{E} \tag{3.2}$$

This means that for $u \in \mathcal{A}^{n-1}, a, b \in \mathcal{A}$ the edge $au \to ub$ corresponds to the word $aub \in \mathcal{A}^{n+1}$. In this way we understand edges in $\mathcal{B}_{\mathcal{A}}(n)$ as words over $\mathcal{A}$ of length $(n+1)$ and vice versa. If we talk about words of length $(n+1)$ as edges of $\mathcal{B}_{\mathcal{A}}(n)$ it is always meant in the way described above.

Let $e = aub \in \mathcal{A}^{n+1}$ be an edge in $\mathcal{B}_{\mathcal{A}}(n)$ and $e = aub$ with $a, b \in \mathcal{A}, u \in \mathcal{A}^{n-1}$. Then au is the initial vertex of e and ub is the terminal vertex of e. Let $\mathcal{E} \subseteq \mathcal{A}^{n+1}$ a set of edges of $\mathcal{B}_{\mathcal{A}}(n)$. Then it follows that

$\mathcal{E}\mathcal{A}^{-1}$	is the set of initial vertices of edges in $\mathcal{E}$,
$\mathcal{A}^{-1}\mathcal{E}$	is the set of terminal vertices of edges in $\mathcal{E}$.

Walks in $\mathcal{B}_{\mathcal{A}}(q)$

Let $e_1, e_2 \in \mathcal{A}^{n+1}$, with $e_1 = aw$ and $e_2 = w'b$. We can understand e_1, e_2 as edges in $\mathcal{B}_{\mathcal{A}}(n)$, as vertices in $\mathcal{B}_{\mathcal{A}}(n+1)$ or as vertices in $L\mathcal{B}_{\mathcal{A}}(n)$. It follows that:

$$
\begin{array}{ll}
 & (e_1, e_2) \text{ is an edge in } L\mathcal{B}_{\mathcal{A}}(n) \\
\Leftrightarrow & \text{The sequence } e_1e_2 \text{ of edges in } \mathcal{B}_{\mathcal{A}}(n) \text{ is a walk of length 2.} \\
\Leftrightarrow & \text{The terminal vertex of the edge } e_1 \text{ in } \mathcal{B}_{\mathcal{A}}(n) \text{ is equal to the} \\
 & \text{initial vertex of the edge } e_2 \text{ in } \mathcal{B}_{\mathcal{A}}(n). \\
\Leftrightarrow & w = w' \\
\Leftrightarrow & e_1 \xrightarrow{b} e_2 \text{ is an edge in } \mathcal{B}_{\mathcal{A}}(n+1).
\end{array}
$$

Therefore we obtain $L\mathcal{B}_{\mathcal{A}}(n) \cong \mathcal{B}_{\mathcal{A}}(n+1)$. By induction follows:

$$L_k\mathcal{B}_{\mathcal{A}}(n) \cong \mathcal{B}_{\mathcal{A}}(n+k) \quad \text{for all } n, k \in \mathbb{N}_0\,. \tag{3.3}$$

The vertex set of $\mathcal{B}_{\mathcal{A}}(n+k)$ is $\mathcal{A}^{n+k}$ and the vertices of $L_k\mathcal{B}_{\mathcal{A}}(n)$ are the walks of length k in $\mathcal{B}_{\mathcal{A}}(n)$. It follows that there exists a one-to-one relation between $\mathcal{A}^{n+k}$ and walks of length k in $\mathcal{B}_{\mathcal{A}}(n)$. Therefore we can interpret words of length $(n+k)$ as walks of length k in $\mathcal{B}_{\mathcal{A}}(n)$.

Let $w = w_1 \dots w_{n+k} \in \mathcal{A}^{n+k}$, with $w_1, \dots, w_{n+k} \in \mathcal{A}$. An easy inductive proof shows, that the walk in $\mathcal{B}_{\mathcal{A}}(n)$ which corresponds to w is given by:

$$w_1 \dots w_n \overset{w_{n+1}}{\rightarrow} w_2 \dots w_{n+1} \overset{w_{n+2}}{\rightarrow} \dots \overset{w_{n+k}}{\rightarrow} w_{k+1} \dots w_{n+k}\,. \tag{3.4}$$

In common we don't make a distinction between a word $w \in \mathcal{A}^{n+k}$ and its corresponding walk of length k in $\mathcal{B}_{\mathcal{A}}(n)$. With (3.4) the next proposition is obviously.

Proposition 29 *Let P be a length k walk in $\mathcal{B}_{\mathcal{A}}(n)$ and $w = w_1 \ldots w_{k+n} \in \mathcal{A}^{n+k}$ the corresponding word, where $w_1, \ldots, w_{k+n} \in \mathcal{A}$.*

(i) *P is a closed walk if and only if $w_1 \ldots w_n = w_{k+1} \ldots w_{n+k}$.*

(ii) *P is a path if and only if w has k different subwords of length $(n+1)$ This means $w_{i+1} \ldots w_{i+n+1} \neq w_{j+1} \ldots w_{j+n+1}$ for all $0 \leq i < j \leq k-1$.*

(iii) *P is a simple path, but is not a cycle, if and only if w has $k+1$ different subwords of length n. This means $w_{i+1} \ldots w_{i+n} \neq w_{j+1} \ldots w_{j+n}$ for all $0 \leq i < j \leq k$*

(iv) *P is a cycle if and only if the word $w_1 \ldots w_{n+k-1}$ has k different subwords of length n and $w_1 \ldots w_n = w_{k+1} \ldots w_{n+k}$.*

(v) *Let $0 \leq m \leq n+k$ and P_m be the walk of length length $(n+k-m)$ in $\mathcal{B}_{\mathcal{A}}(n+m)$ which corresponds to w. If we identify $L_m \mathcal{B}_{\mathcal{A}}(n)$ with $\mathcal{B}_{\mathcal{A}}(n+m)$, it follows that $L_m P = P_m$.*

(vi) *Let P be a closed walk, $m \in \mathbb{N}_0$.Let $p \in \mathbb{N}_0, 0 \leq l < (n+k)$ such that $m = p(n+k) + l$. Then the word $(w_1 \ldots w_{n+k})^{p+1} w_1 \ldots w_l \in \mathcal{A}^{n+k+m}$ corresponds to the walk $\hat{L}_m P$ of length k in $\mathcal{B}_{\mathcal{A}}(n+k)$, where we identify $L_m \mathcal{B}_{\mathcal{A}}(n)$ with $\mathcal{B}_{\mathcal{A}}(n+m)$.*

Cyclic sequences and closed walks in $\mathcal{B}_{\mathcal{A}}(n)$

For a word $w \in \mathcal{A}^n$ with $w = w_0 \ldots w_{n-1}$, $w_i \in \mathcal{A}$, $\forall 0 \leq i < n$, the *cyclic sequence of length n*

$$[w] = [w_0 \ldots w_{n-1}]$$

is defined as the map $w_l : \mathbb{Z} \to \mathcal{A}$ given by

$$w_l := w_{l \bmod n} \quad \forall\, l \in \mathbb{Z} \quad ; \quad [w] = \ldots w_0 \ldots w_{n-1} w_0 \ldots w_{n-1} \ldots \quad .$$

If we work with cyclic n length sequences, we take all subscripts $\bmod n$ without writing this explicitly every time. This means we write w_l in place of $w_{l \bmod n}$ for all $l \in \mathbb{Z}$.

For $t \in \mathbb{Z}, n \in \mathbb{N}$ and $u \in \mathcal{A}^n$ we define:

$$\begin{array}{lcl} \mathrm{Num}_w(u) & := & |\{l \in \mathbb{N}_0 | u = w_l \dots w_{l+n-1}\,,\, 0 \le l < n\}|\,, \\ \mathrm{Sub}_w(n) & := & \{u \in \mathcal{A}^n | u = w_l \dots w_{l+n-1}\,,\, l \in \mathbb{Z}\}\,, \\ [w]_t & := & [w_t \dots w_{t+n-1}]\,, \end{array}$$

then $\mathrm{Sub}_w(n) \subseteq \mathcal{A}^n$ is the set of subwords of length n in the cyclic sequence $[w]$, $\mathrm{Num}_w(u)$ is the total number of occurrence of the word u as a subword in $[w]$ and $[w]_t$ is the the t-shift of the cyclic sequence $[w]$. Obviously $[w]_t = [w]_s$ for $s = t \bmod n$ and $[w] = [w]_t$ if $t = sn$ for some $s \in \mathbb{Z}$.

We call a word $w \in \mathcal{A}^*$ *primitive* , if it is not the power of some other word in $\mathcal{A}^*$. This means

$$w \neq u^n \quad \forall\, u \in \mathcal{A}^* - \{w\},\, n \in \mathbb{N}\,.$$

A cyclic sequence $[w]$ is called primitive, if $w \in \mathcal{A}^+$ is primitive.

Proposition 30 *Let $w \in \mathcal{A}^n$ for some $n \in \mathbb{N}$. The following conditions are equivalent.*

(a) w is primitive.

(b) $[w]_t$ is primitive for all $t \in \mathbb{Z}$.

(c) $[w] \neq [w]_t$ for all $1 \le t < n$

(d) $|\,Sub_w(n)| = n$

We omit a proof of this proposition.

Let us show, that there exists a one-to-one relation between closed pathes of length k in $\mathcal{B}_q(n)$ and cyclic sequences of length k. Let $v = v_0 \dots v_{n+k-1} \in \mathcal{A}^{n+k}$ with $v_i \in \mathcal{A}$ for all $0 \le i \le n+k-1$ .Furthermore let $p \in \mathbb{N}_0$ and $l \in \{0, \dots, k-1\}$ be the unique numbers with $(n-1) = pk + l$.

$$\begin{array}{ll} & v_0 \dots v_{n+k-1} \text{ denotes a closed walk of length } k \text{ in } \mathcal{B}_q(n) \\ \Leftrightarrow & v_0 \dots v_{n-1} = v_k \dots v_{n+k-1} \\ \Leftrightarrow & \qquad (v_0 \dots v_{k-1})(v_k \dots v_{2k-1}) \dots (v_{(p-1)k} \dots v_{pk-1})(v_{pk} \dots v_{pk+l}) \\ & \;\; = \;\; (v_k \dots v_{2k-1}) \dots (v_{(p-1)k} \dots v_{pk-1})(v_{pk} \dots v_{(p+1)k-1})(v_{(p+1)k} \dots v_{(p+1)k+l}) \\ \Leftrightarrow & \left\{ \begin{array}{rclcl} v_0 \dots v_{k-1} & = & v_0 \dots v_{k-1} & = & v_{ik} \dots v_{(i+1)k-1} \\ v_0 \dots v_l & = & v_0 \dots v_l & = & v_{ik} \dots v_{ik+l} \end{array} \right. \quad \forall i \in \{0, \dots, p\} \\ \Leftrightarrow & v = v_{0 \bmod k} v_{1 \bmod k} \dots v_{n\,+\,k\,-\,1 \bmod k} \end{array}$$

This shows, that we can interpret the closed walk P and its corresponding word v as the cyclic sequence $[v_0 \dots v_{k-1}]$. Thus there exists a one-to-one relation between walks of length k in $\mathcal{B}_{\mathcal{A}}(n)$, cyclic sequences of length k and words $v = v_0 \dots v_{n+k-1} \in \mathcal{A}^{n+k}$ with $v_0 \dots v_{n-1} = v_k \dots v_{k+n-1}$, as it is shown below:

$$[\,v\,]$$
$$v = v_0 \dots v_{k-1} \in \mathcal{A}^k$$
$$\updownarrow$$
$$v_0 \dots v_{k+n-1}$$
$$v_i \in \mathcal{A} \,\forall\, i\,,\; v_j = v_{j+k} \;\forall\, 0 \leq k < n$$
$$\updownarrow$$
$$P := v_0 \dots v_{n-1} \xrightarrow{v_n} v_1 \dots v_{n-1} \xrightarrow{v_{n+1}} \dots \xrightarrow{v_{n+k-1}} v_k \dots v_{n+n-1}$$
$$P \text{ is a closed walk of length } k \text{ in } \mathcal{B}_{\mathcal{A}}(n)$$

If we talk about cyclic sequences as closed walks, closed pathes or cycles, it is meant in this way.

Let $t \in \mathbb{Z}, w \in \mathcal{A}^k$, $u \in \mathcal{A}^n$ be a vertex in $\mathcal{B}_{\mathcal{A}}(n)$, $v \in \mathcal{A}^{n+1}$ be an edge of $\mathcal{B}_{\mathcal{A}}(n)$ and P be the closed walk of length k in $\mathcal{B}_{\mathcal{A}}(n)$ which corresponds to $[w]$. Then we obtain:

$\mathrm{Sub}_w(n)$ is the set of vertices in P.

$\mathrm{Sub}_w(n+1)$ is the set of edges in P.

The walk P pass the vertex u $\mathrm{Num}_w(u)$ times.

The walk P runs $\mathrm{Num}_w(v)$ times through the edge v.

The closed walk $[w]_t$ differs from P only in the starting vertex. Especially the starting vertex of $[w]_t$ lays t steps forwards in P if $t \geq 0$ and t steps backwards in P if $t < 0$.

Proposition 31 *Let $w \in \mathcal{A}^k$ for some $k \in \mathbb{N}$.*

(i) For all $n \in \mathbb{N}_0$ we have:

$$\begin{aligned} & [w] \ \text{ is a (closed) path in } \mathcal{B}_q(n) \text{ (of length } k\text{)} \\ \Leftrightarrow \ & |\,Sub_w(n+1)| = k \\ \Leftrightarrow \ & \ Num_w(u) \in \{0,1\} \quad \forall\, u \in \mathcal{A}^{n+1} \end{aligned}$$

(ii) For all $n \in \mathbb{N}_0$ we have:

$$\begin{aligned} & [w] \ \text{ is a cycle in } \mathcal{B}_q(n) \text{ (of length } k\text{)} \\ \Leftrightarrow \ & |\,Sub_w(n)| = k \\ \Leftrightarrow \ & \ Num_w(u) \in \{0,1\} \quad \forall\, u \in \mathcal{A}^{n} \end{aligned}$$

(iii) We identify $\mathcal{B}_\mathcal{A}(n+m)$ with $L_m\mathcal{B}_\mathcal{A}(n)$. Let $[w]$ corresponds to the closed walk P in $\mathcal{B}_\mathcal{A}(n)$. Then for every $m \in \mathbb{N}_0$ $[w]$ corresponds also to the closed walk $\hat{L}_m P$ in $\mathcal{B}_\mathcal{A}(n+m)$.

(iv) We identify $\mathcal{B}_\mathcal{A}(n+1)$ with $L\mathcal{B}_\mathcal{A}(n)$. Let $[w]$ corresponds to the closed walk P in $\mathcal{B}_\mathcal{A}(n)$ and to the closed walk P' in $\mathcal{B}_\mathcal{A}(n+1)$. P' is a cycle if and only if P is a (closed) path.

(v) If there exists $n \in \mathbb{N}$ such that $[w]$ is a closed path in $\mathcal{B}_q(n)$, then $[w]$ is a primitive cyclic sequence.

Proof: (i),(ii),(iii) and (iv) follows from Proposition 29 and the one-to-one corresponding between cyclic sequences of length k and words of length $n+k$. Therefore we have only to show (v). Let $[w] = [w_0 \dots w_{k-1}]$ be a cyclic sequence, such that $[w]$ is a closed path for some $n \in \mathbb{N}$. Let us assume that w is not primitive. Then there are $u \in \mathcal{A}^+$ and $p \geq 2$ with $w = u^p$. It follows:

$$w_0 \dots w_n = w_{|u|} \dots w_{|u|+n} = w_{2|u|} \dots w_{2|u|+n} = \dots = w_{(p-1)|u|} \dots w_{(p-1)|u|+n-1} \,.$$

Since $(p-1)|u| < |w| = k$, we obtain that $\mathrm{Num}_w(w_0 \dots w_n) \geq p \geq 2$. This is a contradiction, because from (i) follows that $\mathrm{Num}_w(v) \leq 1$ for all $v \in \mathcal{A}^{n+1}$. **q.e.d**

3.3 Cycles and 1-factors of $\mathcal{B}_q(n)$

In the rest of this chapter we suppose that $|\mathcal{A}| = q < \infty$. Then $\mathcal{B}_{\mathcal{A}}(n)$ is a finite connected q-regular graph with q^n vertices and q^{n+1} edges. Since $\mathcal{B}_{\mathcal{A}}(n) \cong \mathcal{B}_q(n)$, it is sufficient to restrict our considerations to the graphs $\mathcal{B}_q(n)$. Furthermore we identify $\mathcal{B}_q(n+m)$ with $L_m\mathcal{B}_q(n)$ without writing this explicitly.

Proposition 32 *For every $n \in \mathbb{N}_0, 1 \leq k \leq q$, $\mathcal{B}_q(n)$ contain an Euler circuit, a Hamilton circuit and a k-factor.*

Proof: Obviously $\mathcal{B}_q(n)$ contain an Euler circuit, because $\mathcal{B}_q(n)$ is q-regular, i.e. an Euler graph. Since $\mathcal{B}_q(n)$ is finite and q-regular, it follows from Proposition 26 that there exists a k-factor in $\mathcal{B}_q(n)$. Any loop in $\mathcal{B}_q(0)$ is an Euler and a Hamilton circuit. While $\mathcal{B}_q(n) \cong L\mathcal{B}_q(n-1)$, it follows from the remarks at the end of Proposition 28, that $\mathcal{B}_q(n)$ has a Hamilton circuit for $n \geq 1$.Especially if E is an Euler circuit in $\mathcal{B}_q(n-1)$, then $\hat{L}E$ is a Hamilton circuit in $\mathcal{B}_q(n)$. **q.e.d**

Let $\mathcal{A} = \{0, 1, 2, \ldots, q-1\}$. ¿From Proposition 31 (ii) follows, that a cyclic sequence $[w]$ of length q^n is a Hamilton circuit in $\mathcal{B}_q(n)$ if and only if $\mathrm{Sub}_w(n) = q^n$. This means every word of $\mathcal{A}^n$ occurs exactly one time as a subword in $[w]$. Such sequences are known as q-ary de Bruijn sequences. From the theorem above follows, that there are q-ary de Bruijn sequences of length q^n for every $q \geq 2$ and $n \in \mathbb{N}$. The next theorem answer the question of their number.

Theorem 10 *There exist $((q-1)!)^{q^{n-1}} \cdot q^{q^{n-1}-n}$ q-ary de Bruijn sequences of length q^n and this is also the number of Hamilton circuits in $\mathcal{B}_q(n)$(, where we do not distinguish between cycles and sequences which differs only in the starting vertex.*

The theorem was first shown by de Bruijn [29], Good [30] and Flye-Saint Marie [31] independently. A proof for the binary case can also be found in [28]. An overview of de Bruijn sequences, their history and their constructions ca be found in [29],[35], [36], and [22]. However, in this section we focus on cycles of arbitrary length in $\mathcal{B}_q(n)$. We show, that there exists cycles in $\mathcal{B}_q(n)$ of length L, for every $1 \leq L \leq q^n$. We give two different proofs. First we construct cycles of arbitrary lengths in $\mathcal{B}_2(n)$ with the help of a maximal linear cycle. This construction was given by Golomb in [22]. Then we show in the general case, that there exist cycles of arbitrary length in $\mathcal{B}_q(n)$. This was first shown by Lempel in [23]. Indeed Lempel's proof of the q-ary case does not give a construction of the cycles.

Successor maps of 1-factors of $\mathcal{B}_q(n)$

Let Λ be a 1-factor of $\mathcal{B}_q(n)$, then Λ is the union of vertex disjoint cycles $P_1, \ldots, P_k$. We denote with $L_1, \ldots, L_k$ the lengths of these cycles. We can understand the cycles as (primitive) cyclic sequences, $[w_1], \ldots, [w_k]$ with $w_i \in \mathcal{A}^{L_i}$ for $1 \leq i \leq k$. Let $w_i = w_{0,i} \ldots w_{L_i-1,i}$ for all $1 \leq i \leq k$, where $w_{0,i}, \ldots, w_{L_i-1,i} \in \mathcal{A}$. Every t-shift $[w_i]_t$ of the cycle $[w_i]$, differs from $[w_i]$ only in its starting and end vertex.

Since every vertex of $\mathcal{B}_q(n)$ lays on a unique cycle of the 1-factor, it follows, that $\mathrm{Sub}_{w_1}(n), \ldots, \mathrm{Sub}_{w_k}(n)$ is a partition of $\mathcal{A}^n$. This means, that every word in $\mathcal{A}^n$ is a subword of a unique sequence $[w_1], \ldots, [w_k]$. Let us suppose, that $v \in \mathcal{A}^n$ is a subword of $[w_i]$ for some unique $i \in \{1, \ldots, k\}$. While $[w_i]$ denotes a cycle in $\mathcal{B}_q(n)$, it follows that $\mathrm{Num}_w(v) = 1$. Therefore we obtain an unique $0 \leq j < L_i$ with $v = w_{j,i} \ldots w_{j+n-1,i}$. Thus we can define a map $F : \mathcal{A}^n \to \mathcal{A}$ by:

$$F(w_{j,i} \ldots w_{n+j-1,i}) := w_{n+j,i} \text{ for all } 1 \leq i \leq k \,,\; 0 \leq j < L_i \tag{3.5}$$

Then for every $v := \mathcal{A}^n$ the vertex $v_2 \ldots v_n F(v)$ is the unique successor vertex of v in Λ, where $v = v_1 \ldots v_n$ and $v_1, \ldots, v_n \in \mathcal{A}$.

In the same way we can define a map $\tilde{F} : \mathcal{A}^n \to \mathcal{A}$ such that $\tilde{F}(v)v_1 \ldots v_{n-1}$ is the unique antecessor vertex of v in Λ. This map is given by:

$$\tilde{F}(w_{j,i} \ldots w_{n+j-1,i}) := w_{j-1,i} \text{ for all } 1 \leq i \leq k \,,\; 0 \leq j < L_i \tag{3.6}$$

We call F the *successor map* and $\tilde{F}$ the *antecessor map* of Λ. These maps have the properties:

$$\text{For every } u \in \mathcal{A}^{n-1} \text{ the map } F_u : \mathcal{A} \to \mathcal{A} \text{ which is given by } F_u(a) := F(au) \text{ is a permutation of } \mathcal{A}\,. \tag{3.7}$$

$$\text{For every } u \in \mathcal{A}^{n-1} \text{ the map } \tilde{F}_u : \mathcal{A} \to \mathcal{A} \text{ which is given by } \tilde{F}_u(a) := F(ub) \text{ is a permutation of } \mathcal{A}\,. \tag{3.8}$$

We show only (3.7), because (3.8) follows the same way. If $u \in \mathcal{A}^{n-1}$, $a, b \in \mathcal{A}$, then the vertex $au \in \mathcal{A}^n$ is the antecessor vertex of $uF_u(a) \in \mathcal{A}^n$ in Λ. Since the antecessor vertex is unique, it follows that $uF_u(a) \neq uF_u(b)$ for $a \neq b$. Therefore F_u is a one-to-one map. While F_u is defined for all $a \in \mathcal{A}$, it follows that $\mathcal{F}_u$ is a permutation of $\mathcal{A}$.

Proposition 33 *$\mathcal{F} : \mathcal{A}^n \to \mathcal{A}$ is a successor map of a (unique) 1-factor of $\mathcal{B}_q(n)$ if and only if F fulfill (3.7).*

Proof: We have shown already, that a successor map of a 1-factor has property (3.7). Let $F : \mathcal{A}^n \to \mathcal{A}$ be a map which fulfill property (3.7). Let $v \in \mathcal{A}^n$ we define the sequence $(w_l)_{l\in\mathbb{N}_0}$ by:

$$w_0 \dots w_{n-1} := v \text{ and by induction } w_l := F(w_{l-n} \dots w_{l-1}) \text{ for } l \geq n\,. \tag{3.9}$$

We claim:

$$(w_l)_{l\in\mathbb{N}_0} \text{ is perodical }. \tag{3.10}$$

While $|\mathcal{A}| < \infty$, there are $i < j$ such that $w_i \dots w_{i+n-1} = w_j \dots w_{j+n-1}$. For $i > 0$ and $u := w_i \dots w_{i+n-2}$ we get $F_u(w_{i-1}) = w_{i+n-1} = w_{j+n-1} = F_u(w_{j-1})$. Since F_u is a permutation of $\mathcal{A}$, we obtain $w_{i-1} = w_{j-1}$. It follows that $w_{i-1} \dots w_{i+n-2} = w_{j-1} \dots w_{j+n-2}$ and by induction we find an $L_v \in \mathbb{N}$ with

$$v = w_0 \dots w_{n-1} = w_{L_v} \dots w_{L_v+n-1}\,.$$

Let us choose L_v minimal, then by definition of $(w_l)_{l\in\mathbb{N}_0}$ we have that the sequence is periodical with period L_v. This shows (3.10).

Let $w^v := w_0 \dots w_{L_v-1}$. By (3.10) we obtain, that $[w^v]$ is a cyclic sequence of length L_v. We claim:

$$[w^v] \text{ is a cycle.} \tag{3.11}$$

It is sufficient to show, that

$$|\operatorname{Sub}_{w^v}(n)| = L_v\,.$$

Let us assume, that $|\operatorname{Sub}_{w^v}(n)| < L_v$. Then there are $0 \leq i < j < L_v$ with $w_i \dots w_{i+n-1} = w_j \dots w_{j+n-1}$ with the same argumentation as above we obtain that

$$w_0 \dots w_n = w_{j-i} \dots w_{j-i+n-1}\,.$$

This is a contradiction, because we have chosen L_v minimal and $(j - i) < L_v$. It follows that $|\operatorname{Sub}_{w^v}(n)| = L_v$. By Proposition 31 follows that $[w^v]$ is a cycle in $\mathcal{B}_q(n)$ of length L_v.

We call $[w^v]$ the cycle which *is generated by* F *and* v. For $w \in \mathcal{A}^L$ with $w = w_0 \dots w_{L-1}$, $w_0, \dots, w_{L-1} \in \mathcal{A}$ we obtain:

$$
\begin{aligned}
& [w] \text{ is the cycle generated by } F \text{ and } v \\
\Leftrightarrow \quad & w_0 \dots w_{n-1} = v \;\text{ and }\; F(w_l \dots w_{l+n-1}) = w_{l+n} \quad \forall\, l \in \mathbb{Z}
\end{aligned}
\tag{3.12}
$$

Let $v, v' \in \mathcal{A}^n$. Let $[w]$ be the cycle which is generated by F and v and let $[w']$ be the cycle which is generated by F and v'. By (3.12) follows, that $[w]$ and $[w']$ have a vertex (a subword of length n) in common if and only if $[w'] = [w]_t$ for some $t \in \mathbb{Z}$. In this case both cycles corresponds to the same subgraph in $\mathcal{B}_q(n)$. Let us denote with $\mathcal{C}_v$ the 1-regular subgraph which corresponds to the cycle generated by F and v. It follows, that $\Lambda := \bigcup_{v \in \mathcal{A}^n} \mathcal{C}_v$ is the union of vertex disjoint cycles and every $v \in \mathcal{A}^n$ is a vertex of Λ. Therefore Λ is a 1-factor of $\mathcal{B}_q(n)$. Obviously Λ has F as its successor map. **q.e.d**

Proposition 33 gives us a one-to-one relation between 1-factors of Λ and maps $F : \mathcal{A}^n \to \mathcal{A}$ with property (3.7). Therefore we call maps with property (3.7) successor maps. For a successor map F we obtain the corresponding 1-factor by taking the union of all cycles generated by F and vertices v of $\mathcal{B}_q(n)$. With the same arguments, it can be shown, that there is also a one-to-one correspondence of 1-factors and maps $\tilde{F} : \mathcal{A}^n \to \mathcal{A}$ with property (3.8). Therefore we call maps with property (3.8) antecessor maps.

Every 1-factor of $\mathcal{B}_q(n)$ can be constructed by choosing for every $u \in \mathcal{A}^{n-1}$ a permutation F_u of $\mathcal{A}$. Then the map given by $F(au) := F_u(a)$ for all $u \in \mathcal{A}^{n-1}, a \in \mathcal{A}$, is a successor map of a (unique) 1-factor in $\mathcal{B}_q(n)$. Since the number of permutations of $\mathcal{A}$ is $q!$ and $|\mathcal{A}^{n-1}| = q^{n-1}$ it follows:

Proposition 34 *There are $q^{q! \cdot (n-1)}$ different 1-factors of $\mathcal{B}_q(n)$.*

One might ask, wether a given cycle in $\mathcal{B}_q(n)$ can be extended to a 1-factor. The next lemma shows, that this is possible for every cycle in $\mathcal{B}_q(n)$.

Lemma 35 *Let Γ be a 1-regular subgraph of $\mathcal{B}_q(n)$. (i.e. Γ is the union of vertex disjoint cycles), then there exists a 1-factor Λ of $\mathcal{B}_q(n)$ which is an extension of Γ. This means $\Gamma \subseteq \Lambda$.*

Proof: Let Γ be a 1-regular subgraph of $\mathcal{B}_q(n)$ and let $\mathcal{V}$ be the vertex set of Γ. Then Γ is the vertex disjoint union of some cycles $P_1, \ldots, P_k$. Let $[w_1], \ldots, [w_k]$ be cyclic sequences which corresponds to these cycles. Let L_i be the length of the sequence $[w_i]$ and $w_i = w_{0,i} \ldots w_{L_i-1,i}$ with $w_{0,i}, \ldots, w_{L_i-1,i} \in \mathcal{A}$ for all $1 \leq i \leq k$.

Since the cyclic sequences are vertex disjoint cycles, there are unique $1 \leq j \leq k$ and $0 \leq i \leq L_j$ for every $v \in \mathcal{V}$, such that $v = w_{i,j} \ldots w_{i+n-1,j}$. Thus we can define the map $F : \mathcal{V} \to \mathcal{A}$ as:

$$F(v) := w_{i+n,j} \text{ for all } v \in \mathcal{V} .$$

Let $v = v_1 \ldots v_n \in \mathcal{A}^n$ with $v_1, \ldots, v_n \in \mathcal{A}$. We obtain that $v_2 \ldots v_n F(v) \in \mathcal{V}$ is the unique successor vertex of v in Γ. For $u \in \mathcal{A}^{n-1}$ we define the set $\mathcal{A}_u$ as:

$$\mathcal{A}_u := \{a | v = au, a \in \mathcal{A}, au \in \mathcal{V}\} .$$

If $F(au) = F(bu)$ for some $a, b \in \mathcal{A}_u$ with $a \neq b$, it follows that $au, bu \in \mathcal{V}$ are two different antecessor vertices of the vertex $uF(au) = uF(bu)$ in Γ. This is a contradiction, because Γ is 1-regular. We conclude that the map $F_u : \mathcal{A}_u \to \mathcal{A}$ is a one-to-one map, where F_u is defined for $u \in \mathcal{A}^{n-1}$ as $F_u(a) := F(au)$ for all $a \in \mathcal{A}_u$. (If $au \notin \mathcal{V}$ for all $a \in \mathcal{A}$, then F_u is the empty map.) Since F_u is one-to-one, we can extend F_u to a permutation of $\mathcal{A}$. This gives us an extension of the map $F : \mathcal{V} \to \mathcal{A}$ to a map $F' : \mathcal{A}^n \to \mathcal{A}$. Obviously F' fulfill the property (3.7). Therefore F' is a successor map of some 1-factor Λ. Since F' is an extension of F, it follows by the definition of F, that $\Gamma \subseteq \Lambda$. **q.e.d**

Maximal linear cycles in $\mathcal{B}_2(n)$

A map $F : \mathcal{A}^n \to \mathcal{A}$ is called a *linear map*, if there exists $c_1, \ldots, c_n \in \mathcal{A}$ such that

$$F(w) := c_1 w_1 + \ldots + c_n w_n \bmod q \text{ for } w = w_1 \ldots w_n \in \mathcal{A}^n ,$$

Furthermore the map F is called a *linear successor map*, if c_1 is not a divisor of zero in $\mathbb{Z}_q$. This means $c_1 a \bmod q \neq 0$ for all $a \in \mathcal{A}$. For example $c_1 = 1$ is not a divisor of zero for all $q \geq 2$, but 2 is a divisor of zero in $\mathbb{Z}_4$, since $2 \cdot 2 \bmod 4 = 0$.

Let us show, that every linear successor map F is really a successor map. Let $u := u_1 \dots u_{n-1}$ and $a, b, u_1, \dots u_{n-1} \in \mathcal{A}$, then:

$$\begin{aligned} & F_u(a) = F_u(b) \Leftrightarrow c_1 a + c_2 u_1 + \ldots + c_n u_n \bmod q = c_1 b + c_2 u_1 + \ldots + c_n u_n \bmod q \\ \Leftrightarrow \quad & c_1 a \bmod q = c_1 b \bmod q \\ \Leftrightarrow \quad & c_1 (a - b) \bmod q = 0 \,. \end{aligned}$$

Since c_1 is not a divisor of zero, by the last equation follows:

$$(a - b) \bmod q = 0 \Leftrightarrow a = b \,.$$

We conclude that F has property (3.7), i.e. F is a successor map of some 1-factor of $\mathcal{B}_q(n)$.

If F is a linear successor map, then the cycle which is generated by F and 0^n is the loop at the vertex 0^n. Therefore $q^n - 1$ is the maximal possible length of a cycle $[w]$ which is generated by a linear successor map F and a vertex $v \in \mathcal{A}^n$. If $[w]$ is of maximum length $q^n - 1$, then every word in $\mathcal{A}^n - \{0^n\}$ is a subword of $[w]$ and for every word $v' \in \mathcal{A}^n - \{0^n\}$ the cycle $[w']$ which is generated by F and v' is a t-shift of $[w]$ for some $t \in \mathbb{Z}$.

Defenition 3 (and proposition)

(i) *The cyclic sequence* $[w]$ *is called a* maximal linear cycle *in* $\mathcal{B}_q(n)$*, if* $[w]$ *has length* $q^n - 1$ *and is generated by a linear successor map* $F : \mathcal{A}^n \to \mathcal{A}$ *and some* $v \in \mathcal{A}^n$*.*

(ii) *A linear successor map* $F : \mathcal{A}^n \to \mathcal{A}$ *is called* maximal linear (successor) map *, if* F *generates for some* $v \in \mathcal{A}^n$ *a maximal linear cycle.*

(iii) *Let* $[w]$ *be a maximal linear cycle which is generated by* F *and* $v \in \mathcal{A}^n$*. Then* $Sub_w(n) = q^n - 1$*,* $Num_w(0^n) = 0$ *and* $Num_w(u) = 1$ *for all* $u \in \mathcal{A}^n - \{0^n\}$*. If* $[w']$ *is another cyclic sequence, which is generated by* F *and some* $v' \in \mathcal{A}^n - \{0^n\}$*, then* $[w'] = [w]_t$ *for some* $0 \leq t < q^n - 1$*.*

The question rises, wether maximal linear cycles exist in $\mathcal{B}_q(n)$? We will give an answer only for the binary case.

The Euler φ-function is defined as follows:
For $n > 1$, let

$$n = \prod_{i=1}^{m} p_i^{k_i} \text{ with } p_1 < \ldots < p_m \text{ primes, } k_1, \ldots, k_m \in \mathbb{N} \tag{3.13}$$

be the unique factorization of n into primes. Then φ is defined as:

$$\phi(n) := \begin{cases} 1 & \text{if } n = 1 \\ \prod_{i=1}^{m} p_i^{k_i - 1}(p_i - 1) & \text{if } n > 1 \end{cases} \tag{3.14}$$

For $q \geq 2$ the function $\lambda_q : \mathbb{N} \to \mathbb{N}$ is given by:

$$\lambda_q(n) := \frac{\phi(q^n - 1)}{n} \tag{3.15}$$

Theorem 11 (Golomb [22]) *Let $\mathcal{A} := \{0, 1\}$. Then there exists $\lambda_2(n)$ maximal linear maps $F : \mathcal{A}^n \to \mathcal{A}$.*

A proof of the theorem can be found in [22].

Since $\lambda_2(n) \geq 1$ for all $n \in \mathbb{N}$, it follows that there exist maximal linear cycles in $\mathcal{B}_2(n)$ for all $n \in \mathbb{N}$. Furthermore we obtain the following proposition:

Proposition 36 *Let $n \in \mathbb{N}$. There are $\lambda_2(n) \cdot (2^n - 1)$ maximal linear binary cycles and $\mathcal{B}_2(n)$ contains $\lambda_2(n)$ different subgraphs of maximal linear cycles.*

Cycles of arbitrary lengths in $\mathcal{B}_2(n)$ obtained from maximal linear cycles

In this section we give a construction of cycles in $\mathcal{B}_q(n)$ of arbitrary length, by splitting a maximal cycle into two cycles of length L and length $q^n - 1 - L$. The construction was proposed by Golomb in [22]. It works for every maximal cycle of $\mathcal{B}_q(n)$. Therefore every maximal cycle in $\mathcal{B}_q(n)$ gives us cycles of length L in $\mathcal{B}_q(n)$ for every $1 \leq L \leq q^n - 1$. Since there exists maximal linear cycles in $\mathcal{B}_2(n)$ for every $n \in \mathbb{N}$, it follows that there exist cycles of length L in $\mathcal{B}_2(n)$ for every $1 \leq L \leq 2^n - 1$. While $10^{n-1} \to 0^{n-1}1$ is an edge of every maximal linear cycle in $\mathcal{B}_q(n)$, we obtain an Hamilton circuit for $\mathcal{B}_q(n)$ by replacing the edge $10^n \to 0^n1$ of a maximal linear cycle with the path $10^{n-1} \to 0^n \to 0^{n-1}1$. Therefore we can obtain cycles of arbitrary lengths in $\mathcal{B}_q(n)$ from only one maximal linear cycle of $\mathcal{B}_q(n)$. Constructions of maximal linear cycles in $\mathcal{B}_q(n)$ with primitive polynomials ca be found in [22].

Let $F : \mathcal{A}^n \to \mathcal{A}$ be a maximal linear successor map.

$$\begin{aligned} &F(v_1 \dots v_n) := c_1 v_1 + \dots c_n v_n \quad \text{for all } \ v_1, \dots, v_n \in \mathcal{A} \ \text{ with} \\ &c_1, \dots, c_n \in \mathcal{A} \ \text{ such that } c_1 \ \text{ is not a divisor of zero in } \mathbb{Z}_q\,. \end{aligned} \tag{3.16}$$

The cyclic sequence $[w]$ which is generated by F and $10^{n-1} \in \mathcal{A}^n$ is a maximal linear cycle, i.e. $|w| = |\operatorname{Sub}_w(n)| = q^n - 1\,,$ $\operatorname{Sub}_w(n) = \mathcal{A}^n - \{0^n\}$ and every word in $\mathcal{A}^n - \{0^n\}$ occurs exactly one time as a subword in $[w]$.

$[w]_t$ is for all $t \in \mathbb{Z}$ a maximal linear cycle, too. It differs from $[w]$ only in the start- and end vertex. Let $v \in \mathcal{A}^n - \{0^n\}$, then there exists a unique t with $0 \leq t < q^n - 1$ such that the cycle which is generated by F and v is the maximal linear cycle $[w]_t$.

Let

$$[\mathbf{0}] := [0^{q^n - 1}] \ \text{ and } \ \mathcal{G} := \{[\mathbf{0}], [w], [w]_1, \dots, [w]_{q^n-2}\}\,. \tag{3.17}$$

Then $|\mathcal{G}| = q^n$. We define the binary operations $\oplus$ and $\ominus$ on $\mathcal{G}$ by:

$$\begin{aligned} &\text{For } [w'], [w''] \in \mathcal{G} \ \text{ with } w' = w'_0 \dots w'_{q^n-2}\,,\ w'' = w''_0 \dots w''_{q^n-2}, \\ &w'_l, w''_l \in \mathcal{A} \ \text{ for all } \ 0 \leq l < q^n - 1 : \\ &[w'] \oplus [w''] := [u_0 \dots u_{q^n-2}] \text{ with } u_l := (w'_l + w''_l) \bmod q \ \text{ for } 0 \leq l < q^n - 1 \\ &[w'] \ominus [w''] := [u_0 \dots u_{q^n-2}] \text{ with } u_l := (w'_l - w''_l) \bmod q \ \text{ for } 0 \leq l < q^n - 1 \end{aligned} \tag{3.18}$$

This means, $\oplus$ is the componentwise addition mod q of cyclic sequences.

Lemma 37 *$(\mathcal{G}, \oplus, [\mathbf{0}])$ is an Abelian group with identity $[\mathbf{0}]$ and $[\mathbf{0}] \ominus [w']$ is the negative element of $[w']$ for all $[w'] \in \mathcal{G}$, i.e. $([w'] \oplus [w''])\,,\, ([w'] \ominus [w'']) \in \mathcal{G}$ for all $[w'], [w''] \in \mathcal{G}$*

Proof: Let $w = w_0 \dots w_{q^n-2}$ with $w_0, \dots, w_{q^n-2} \in \mathcal{A}$. We have to show:

(i) $([w'] \oplus [w'']) \oplus [w'''] = [w'] \oplus ([w''] \oplus [w'''])$ for all $[w'], [w''], [w'''] \in \mathcal{G}$,

(ii) $[w'] \oplus [w''] = [w''] \oplus [w']$ for all $[w'], [w''] \in \mathcal{G}$,

(iii) $[w'] \oplus [\mathbf{0}] = [w']$ for all $[w'] \in \mathcal{G}$,

(iv) For all $[w'] \in \mathcal{G}$ there is $[w''] \in \mathcal{G}$ with $[w''] = [\mathbf{0}] \ominus [w']$ and $[w'] \oplus [w''] = [w'] \ominus [w'] = [\mathbf{0}]$,

(v) $[w'] \oplus [w''] \in G$ for all $[w'], [w''] \in \mathcal{G}$.

Obviously (i)-(iii) hold by the definition of $\oplus$ and (iv) holds for $[w'] = [\mathbf{0}]$.Let $[w'] = [w]_t$ for some $0 \le t < q^n - 1$. Let $w' = w'_0 \dots w'_{q^n-2} = w_t \dots w_{t+q^n-1}$, $w''_l := (-w'_l) \bmod q$ for all l and

$$[w''] := [\mathbf{0}] \ominus [w'] = [w''_0 \dots w''_{q^n-2}]\,.$$

We obtain $(w''_0 \dots w''_{n-1})\,,\, (w'_0 \dots w'_{n-1}) \neq 0^n$, because $\mathrm{Sub}_w(n) = \mathcal{A}^n - \{0^n\}$.

Since $F : \mathcal{A}^n \to \mathcal{A}$ is linear and $[w']$ is generated by F and $w'_0 \dots w'_{n-1}$ we obtain for all $l \in \mathbb{Z}$:

$$\begin{aligned} 0 &= F\Big((w'_l + w''_l \bmod q) \dots (w'_{l+n-1} + w''_{l+n-1} \bmod q)\Big) \\ &= F(w'_l \dots w'_{l+n-1}) + F(w''_l \dots w''_{l+n-1}) \bmod q \\ &= w'_{l+n} + F(w''_l \dots w''_{l+n-1}) \bmod q\,. \end{aligned}$$

It follows that:

$$F(w''_l \dots w''_{l+n-1}) = (-w'_{l+n}) \bmod q = w''_{l+n} \quad \forall\, l \in \mathbb{Z}\,.$$

The cyclic sequence generated by F and $w''_0 \dots w''_{n-1}$ satisfies the above equation as well. Since F is maximal and $w''_0 \dots w''_{n-1} \neq 0$, the sequence has length $q^n - 1$. Therefore $[w'']$ is the maximal linear cycle generated by F and $w''_0 \dots w''_{n-1}$. While $\mathrm{Sub}_w(n) = \mathcal{A}^n - \{0^n\}$, there exists a (unique) $s \in \{0 \dots, q^n - 2\}$ such that $w''_0 \dots w''_{n-1} = w_s \dots w_{s+n-1}$. Thus we obtain $[w''] = [w]_s \in \mathcal{G}$. By the definition of $[w'']$ it follows that $[w'] \oplus [w''] = [w'] \ominus [w'] = [\mathbf{0}]$. This shows (iv).

Let $[w'], [w''] \in G$. If one of the cyclic sequences is equal to $[\mathbf{0}]$, then (v) is obviously true. Thus let $[w'], [w''] \neq [\mathbf{0}]$. Then both sequences are generated by F and their first n terms. Let $[v_0 \dots v_{q^n-2}] := [w'] \oplus [w'']$ Since F is linear, it follows:

$$F(v_l \dots v_{l+n-1}) = v_{l+n} \quad \forall\, l \in \mathbb{Z}\,. \tag{3.19}$$

Thus $[v_0 \dots v_{q^n-2}] = [\mathbf{0}]$ if and only if $v_0 \dots v_{n-1} = 0^n$. In this case we have $[w'] \oplus [w''] \in \mathcal{G}$. Let us suppose that $v_0 \dots v_{n-1} \neq 0^n$. While F is maximal the cyclic sequence which is generated by F and $v_0 \dots v_{n-1}$ also satisfy (3.19), i.e. this sequence has length $q^n - 1$. Therefore $[w'] \oplus [w'']$ is the cyclic sequence which is generated by F and $v_0 \dots v_{n-1}$. Since $\mathrm{Sub}_w(n) = \mathcal{A}^n - \{0^n\}$, we obtain that $v_0 \dots v_{n-1} = w_t \dots w_{t+n-1}$ for some (unique) $0 \leq t < q^n - 1$. It follows that $[w'] \oplus [w''] = [v_0 \dots v_{q^n-2}] = [w]_t \in \mathcal{G}$. This shows (v). **q.e.d**

Let $\Gamma := (\mathcal{V}, \mathcal{E})$ be a digraph without multiple edges. Let P be a cycle in Γ. Since Λ has no multiple edges, we can interpret P as a sequence $v_1 \dots v_n$ of vertices with $v_i \in \mathcal{V}$ for all $1 \leq i \leq n$. P can be *split into two cycles* P_1, P_2, if there exists i,i $1 \leq i < j \leq n$ such that

$$P_1 = v_1 \dots v_i v_{j+1} \dots v_{n+1}\,, \quad P_2 = v_{i+1} \dots v_j v_{i+1}\,.$$

This is possible if and only if $v_i \to v_{j+1}$ and $v_j \to v_{i+1}$ are edges of Γ. Obviously these edges are not contained in the cycle P.

Theorem 12 (Goloumb [22]) *Let $2 \leq L \leq q^n - 2$. Every maximal linear cycle in $\mathcal{B}_q(n)$ can be split into two cycles of length L and $q^n - L - 1$.*

Proof : Let $[w]$ be a maximal linear cycle in $\mathcal{B}_q(n)$, where $w = w_0 \dots w_{q^n-2}$ with $w_0, \dots, w_{q^n-2} \in \mathcal{A}$. Let $v \in \mathcal{A}^n$ and $F : \mathcal{A}^n \to \mathcal{A}$ be a maximal linear map, such that $[w]$ is generated by F and v.

$$\begin{array}{ll} F(u_1 \dots u_n) := c_1 u_1 + \dots c_n u_n \bmod q & \text{for all } u_1, \dots, u_n \in \mathcal{A} \\ \text{with } c_1, \dots, c_n \in \mathcal{A} \text{ and } c_1 \neq 0 \text{ is not a divisor of zero in } \mathbb{Z}_q & \end{array}\,.$$

Since F is maximal, the cycle which is generated by F and $10^{n-1} \in \mathcal{A}^n$, is a t-shift of $[w]$ for some $0 \leq t < q^n - 1$, i.e. $[w]$ and the cycle which is generated by F and 10^{n-1} have the same subgraph in $\mathcal{B}_q(n)$. Therefore it is sufficient to show the theorem for $w_0 \dots w_{n-1} = 10^{n-1}$.

Thus let $w_0 \dots w_{n-1} = 10^{n-1}$. Let $\oplus, \ominus$ be as in Lemma 37 the componentwise addition and substraction modulo q of cyclic sequences. We have $[w] \neq [w]_L$ for all L with $1 \leq L < q^n - 1$, because $[w]$ is a cycle in $\mathcal{B}_q(n)$. Therefore

$$w_0 \dots w_{n-1} \neq w_L \dots w_{L+n-1}\,.$$

By Lemma (37) follows, that for every $1 \leq L < q^n - 1$ there is a unique $0 \leq M < q^n - 1$ such that:

$$[w] \ominus [w]_L = [w]_M \Rightarrow [w] = [w]_L \oplus [w]_M \,. \tag{3.20}$$

Since $[w]_M = [w_M \dots w_{M+q^n-2}]$ is also maximal linear, it follows that every word in $\mathcal{A}^n - \{0^n\}$ occurs as a subword in $[w]_M$. Thus, for every $1 \leq a \leq (q-1)$ there exists an m with $0 \leq m \leq (q^n - 2)$, such that $w_{m+M} \dots w_{m+M+n-1} = 0 \dots 0a$. With (3.20) follow:

$$\begin{aligned}
w_m &= w_{m+L}\,, \\
w_{m+1} &= w_{m+L+1}\,, \\
&\vdots \quad\quad , \\
w_{m+n-2} &= w_{m+L+n-2}\,, \\
w_{m+n-1} &= w_{m+L+n-1} + a \bmod q\,.
\end{aligned}$$

It follows that we can split the maximal linear cycle $[w]$ into two cycles of length L and $(q^n - 1 - L)$. These cycles are given by:

Cycle of length L:

$$\begin{aligned}
& w_m \dots w_{m+n-1} \\
\xrightarrow{w_{m+n}} \quad & w_{m+1} \dots w_{m+n} \\
\longrightarrow \quad & \dots \\
\xrightarrow{w_{m+L+n-3}} \quad & w_{m+L-2} \dots w_{m+L+n-3} \\
\xrightarrow{w_{m+L+n-2}} \quad & w_{m+L-1} \dots w_{m+L+n-2} = w_{m+L-1} w_m \dots w_{m+n-2} \\
\xrightarrow{w_{m+n-1}} \quad & w_m \dots w_{m+n-1}
\end{aligned}$$

Cycle of length $(q^n - 1 - L)$:

$$\begin{aligned}
& w_{m+L} \dots w_{m+L+n-1} \\
\xrightarrow{w_{m+L+n}} \quad & w_{m+L+1} \dots w_{m+L+n} \\
\longrightarrow \quad & \dots \\
\xrightarrow{w_{m+q^n+n-3}} \quad & w_{m+q^n-3} \dots w_{m+q^n+n-3} \\
\xrightarrow{w_{m+q^n+n-2}} \quad & w_{m+q^n-2} \dots w_{m+q^n+n-2} = w_{m+q^n-2} w_m \dots w_{m+n-2} \\
& = w_{m+q^n-2} w_{m+L} \dots w_{m+L+n-2} \\
\xrightarrow{w_{m+L+n-1}} \quad & w_{m+L} \dots w_{m+L+n-1}
\end{aligned}$$

q.e.d

Proposition 38 *Let $\mathcal{A} = \{0, 1\}$. There exists a cycle of length L in $\mathcal{B}_2(n)$ for every $1 \leq L \leq 2^n$.*

Proof: By Proposition 36 there exists a maximal linear cycle in $\mathcal{B}_2(n)$. If $2 \leq L \leq 2^n - 1$, it follows from Theorem 12, that there exists a cycle of length L in $\mathcal{B}_2(n)$. If $L = 2^n$, we obtain a cycle of length 2^n by replacing the edge $10^{n-1} \to 0^{n-1}1$ of a maximal linear cycle with the path $10^{n-1} \to 0^n \to 0^{n-1}1$. If $L = 1$, then the loop $0^n \to 0^n$ is a cycle of length 1 in $\mathcal{B}_2(n)$. **q.e.d** .

Cycles of arbitrary lengths in $\mathcal{B}_q(n)$

Theorem 13 (Lempel[23])
Let $1 \leq L \leq q^n$. Then $\mathcal{B}_q(n)$ contains a cycle of length L.

Obviously $\mathcal{B}_q(n)$ contain a cycle of length 1 for every $n \in \mathbb{N}_0$. For example the loop $0^n \xrightarrow{0} 0^n$ is such a cycle. Let $n \in \mathbb{N}$ and Λ be a connected Euler subgraph in $\mathcal{B}_q(n-1)$ with L edges. If E is an Euler circuit for Λ, then from Proposition 28 (iii) follows that $\hat{L}E$ is a cycle of length L in $\mathcal{B}_q(n)$. Therefore it is sufficient to show, that $\mathcal{B}_q(n)$ contain a connected Euler subgraph with L edges for every $1 \leq L \leq q^{n+1}$ and $n \in \mathbb{N}_0$.

Lemma 39 (Lempel) *Let $n \in \mathbb{N}_0$. For every $1 \leq L \leq q^{n+1}$ there exists a connected Euler subgraph in $\mathcal{B}_q(n)$ with L edges.*

Proof: We proof the lemma by induction on n.
n=0
For L with $1 \leq L \leq q$ we can choose L loops in $\mathcal{B}_q(0)$. This gives us a connected Euler subgraph of $\mathcal{B}_q(0)$ with L edges. Therefore the lemma holds for $n = 0$.

n-1$\to$ n
Let us assume that there exists a connected Euler subgraph in $\mathcal{B}_q(n-1)$ with L edges for every $1 \leq L \leq q^n$. Let $1 \leq L \leq q^{n+1}$. We show that there exists a connected Euler subgraph in $\mathcal{B}_q(n)$ with L edges. We distinguish two cases.

Case 1: $L \leq q^n$
By induction hypothesis $\mathcal{B}_q(n-1)$ contains a connected Euler graph with L edges. If P is an Euler circuit for this subgraph, then $\hat{L}P$ is a cycle of length L in $\mathcal{B}_q(n)$, where we identify $\mathcal{B}_q(n)$ with $L\mathcal{B}_q(n-1)$. This follows from Proposition 28 (iii). Since a cycle of length L is a connected 1-regular graph with L edges, we have found a connected Euler subgraph in $\mathcal{B}_q(n)$ with L edges.

Case 2: $q^n < L \leq q^{n+1}$
Let $1 \leq m \leq (q-1)\,,\, 0 < k \leq q^n$ such that $L = mq^n + k$ and $L' := q^n - k < q^n$.

If $L' > 0$, then by the induction hypothesis it follows that $\mathcal{B}_q(n-1)$ contains a connected Euler graph with L' edges. Let E be an Euler circuit in this graph. Then from Proposition 28 (iii) follows that $C := \hat{L}P$ is a cycle of length L' in $\mathcal{B}_q(n)$. If $L' = 0$, then we choose an arbitrary vertex in $\mathcal{B}_q(n)$. This gives us a cycle C of length 0 in $\mathcal{B}_q(n)$. ¿From Lemma 35 follows that there is a 1-factor Γ_1 in $\mathcal{B}_q(n)$ which contains the cycle C. While $m+1 \leq q$ and $\mathcal{B}_q(n)$ is q-regular, from Proposition 26 follows that there exists a $(m+1)$-factor Γ_2 of $\mathcal{B}_q(n)$ with $\Gamma_1 \subseteq \Gamma_2$. Let Γ_3 be the complement of C in Γ_2. We obtain Γ_3 if we delete in Γ_2 all edges of C. Since Γ_2 is $m+1$-regular and $m+1 \geq 2$, it follows, that the graph Γ_3 is a spanning Euler subgraph in $\mathcal{B}_q(n)$ without isolated vertices. Since C is contained in $\Gamma_1 \subseteq \Gamma_2$ and has L' edges, we obtain for the number of edges in Γ_3:

$$(q^n - L') + mq^n = (q^n - (q^n - k)) + mq^n = mq^n + k = L\,.$$

Let $\Lambda_1, \ldots, \Lambda_p$ be the connected components of Γ_3. If $p = 1$, then the proof is finished, because in this case Λ_3 is a connected Euler subgraph of $\mathcal{B}_q(n)$ with L edges.

Thus assume, that $p \geq 2$. While $\mathcal{B}_q(n)$ is connected and Γ_3 is a spanning subgraph of $\mathcal{B}_q(n)$, we conclude, that there is an edge e in $\mathcal{B}_q(n)$ whose initial and terminal vertices are in two different connected components of Γ_3. Let $w_1 \ldots w_{n+1} \in \mathcal{A}^{n+1}$ be the label of e, where $w_1, \ldots, w_{n+1} \in \mathcal{A}$. Then $w := w_1 \ldots w_n$ is the initial vertex of e and $w' := w_2 \ldots w_{n+1}$ is the terminal vertex. Let w contained in Λ_i and w' contained in Λ_j for some i, j with $i < j$. While Γ_3 is an Euler graph without isolated vertices, it follows that there is an edge in Λ_i with initial vertex w and an edge in Λ_j with terminal vertex w'. This means, there are $a, b \in \mathcal{A}$ with $a \neq w_1$ and $b \neq w_{n+1}$, such that $w_1 \ldots w_n b$ is an edge in Λ_i and $a w_2 \ldots w_{n+1}$ is an edge in Λ_j. It follows that $a w_2 \ldots w_n b$ is an edge, different to $w_1 \ldots w_{n+1}$, which also connect Λ_i with Λ_j. Especially the edge $a w_2 \ldots w_n b$ is not an edge in Γ_3. We delete the edges $w_1 \ldots w_n b$ and $a w_2 \ldots w_{n+1}$ in Γ_3 and replace them by the edges $w_1 \ldots w_{n+1}$ and $a w_2 \ldots w_n b$.

Thus we obtain a new spanning subgraph Γ_4 of $\mathcal{B}_q(n)$ with L edges which is also an Euler graph without isolated vertices, but has only $(p-1)$ connected components. Especially the connectivity components of Γ_4 are

$$\Lambda_i \cup \Lambda_j, \Lambda_1, \ldots, \Lambda_{i-1}, \Lambda_{i+1}, \ldots \Lambda_{j-1}, \Lambda_{j+1}, \ldots, \Lambda_p .$$

If we proceed to connect the connectivity components of Γ_3 in this way, we obtain after $(p-1)$ steps a connected spanning Euler subgraph of $\mathcal{B}_q(n)$ with L edges. **q.e.d**

3.4 Regular subgraphs of $\mathcal{B}_q(n)$

In this section we study k-regular subgraphs of $\mathcal{B}_q(n)$, i.e. we deal with the question, wether there exists a k-regular subgraph of $\mathcal{B}_q(n)$ for a given number of vertices. Throughout this section we denote with $\mathcal{A}$ the alphabet $\mathcal{A} = \{0, \ldots q-1\}$ for some $q \geq 2$.

Let $\Lambda \subseteq (\mathcal{V}, \mathcal{E})$ be a subgraph of $\mathcal{B}_q(n)$. If we understand the edges in Λ as words of length $n+1$, then we obtain that Λ is k-regular if and only if there exist for every $v \in \mathcal{V}$ unique sets $\mathcal{B}_a, \mathcal{B}_s \subseteq \mathcal{A}$ such that $\mathcal{B}_a v, v\mathcal{B}_s \subseteq \mathcal{E}$ and $|\mathcal{B}_a| = |\mathcal{B}_v| = k$. In this case $\mathcal{B}_a v$ are the edges of Λ incident to v and $v\mathcal{B}_s$ are the edges of Λ incident from v. Furthermore we obtain that $k|\mathcal{V}| = |\mathcal{E}|$.

Since a cycle of length L is also a connected 1-regular graph with L vertices, we obtain by Lempel's Theorem 39:

Proposition 40 *For every $n \in \mathbb{N}_0$ and $L \in \mathbb{N}$ with $1 \leq L \leq q^n$ there exists a connected 1-regular subgraph in $\mathcal{B}_q(n)$ with L vertices.*

Let $2 \leq k \leq q$. Since $\mathcal{B}_k(n)$ is a k-regular subgraph of $\mathcal{B}_q(n)$ we obtain:

Proposition 41 *Let $1 \leq k \leq q$. For every $n \in \mathbb{N}_0$ there exists a connected k-regular subgraph in $\mathcal{B}_q(n)$ with k^n vertices and k loops.*

Proposition 42 *If there exists k-regular subgraph of $\mathcal{B}_q(n)$ with L vertices and l loops then there exists also a k-regular subgraph in $\mathcal{B}_q(n+m)$ with $L \cdot k^m$ vertices and l loops. This holds also for "connected k-regular subgraph" in place of "k-regular subgraph".*

Proof: Let $\Lambda = (\mathcal{V}, \mathcal{E})$ be a k-regular subgraph of $\mathcal{B}_q(n)$ with L vertices and l loops. By Proposition 27 follows that $L_m\Lambda$ is a k-regular subgraph with $L \cdot k^m$ vertices and l loops. Furthermore we obtain that $L_m\Lambda$ is connected, if Λ is connected. By identifying $L_m\mathcal{B}_q(n)$ with $\mathcal{B}_q(n+m)$ we obtain the proposition. **q.e.d**

Since any connectivity component of a k-factor of $\mathcal{B}_q(n)$ is by itself a connected k-regular subgraph of $\mathcal{B}_q(n)$, we can find k-regular subgraphs of $\mathcal{B}_q(n+m)$ by taking the m-iterated lingraph of connected components of k-factors of $\mathcal{B}_q(n)$. However for $k \geq \lceil \frac{q}{2} \rceil$ every k-factor of $\mathcal{B}_q(n)$ is a connected graph.[1]

We continue with an example of Proposition 42.

Example 12 Let $2 \leq k \leq q$. Let $A \in \mathcal{A}$ be set of letters with $|A| \geq k$ and let $\varphi : \{0, \dots |A|-1\} \leftrightarrow A$ be a bijection. We define the subgraph $\mathcal{N}(A, \varphi, k) = (\mathcal{V}, \mathcal{E}) \subseteq (\mathcal{A}, \mathcal{A}^2)$ in $\mathcal{B}_q(1)$ as follows:

Let $[w]$ be the cyclic sequence

$$[w] := [w_0 \dots w_{|A|-1}] \;\text{ with }\; w_l := \varphi(l) \quad \forall\, 0 \leq l < |A|\,.$$

Since φ is a bijection, it follows that $[w]$ is a cycle in $\mathcal{B}_q(1)$ of length bigger than or equal to k. We define the vertex set $\mathcal{V}$ and the edge set $\mathcal{E}$ of $\mathcal{N}(A, \varphi, k)$ as:

$$\begin{aligned} \mathcal{V} &:= A = \mathrm{Sub}_w(1) \subseteq \mathcal{A}^1\,, \\ \mathcal{E} &:= \{w_l w_{l+i} \,|\, l \in \mathbb{Z}\,, 0 \leq i < k\}\,. \subseteq \mathcal{A}^2 \end{aligned}$$

Every letter in $A = \mathrm{Sub}_w(1)$ occurs exactly one time in w and $|w| = |A| \geq k$. Hence we obtain, that for every $a \in \mathcal{V} = \mathrm{Sub}_w(1)$ there exist k unique letters $b_1, \dots b_k \in A$ with $ab_j \in \mathcal{E}$ and further on k unique letters $c_1 \dots c_k \in A$ with $c_j a \in \mathcal{E}$ for $j \in \{1, \dots, k\}$. Therefore $\mathcal{N}(A, \varphi, k)$ is a k-regular subgraph in $\mathcal{B}_q(1)$ with $|A|$ vertices. Furthermore aa is a loop in $\mathcal{N}(A, \varphi, k)$ for every $a \in A$. Obviously $\mathcal{N}(A, \varphi, k)$ contains no other loops. Finally we obtain that $\mathcal{N}(A, \varphi, k)$ is connected, because $[w]$ is a Hamilton circuit for $\mathcal{N}(A, \varphi, k)$.

Let $0 \leq p \leq q-k$ and $A := \{0, \dots, k+p-1\}$. It follows that $\mathcal{N}(A, id_A, k)$ is a connected k-regular subgraph of $\mathcal{B}_q(1)$ with $k+p$ vertices and a loop at each vertex. Thus we obtain by Proposition 42 the following result:

Proposition 43 *Let $2 \leq k \leq q$ and $0 \leq p \leq q-k$. There exists a connected k-regular subgraph in $\mathcal{B}_q(n)$ with $k^n + p \cdot k^{n-1}$ vertices and $k+p$ loops for every $n \in \mathbb{N}$.*

[1] We omit a prove of this statement.

Cyclic sequences of regular subgraphs of $\mathcal{B}_q(n)$

Let Λ be a k-regular subgraph of $\mathcal{B}_q(n)$ and E be an Euler circuit for Λ. Since Λ has kL edges, the closed path E has length kL. Let $[w]$ be the corresponding cyclic sequence of length kL. Since E is a closed path with L vertices and runs through every of its vertices exactly k times, it follows that $[w]$ has the following properties:

$$\begin{aligned} &\text{(a)} \quad \mathrm{Num}_w(v) = k \ \text{ for all } \ v \in \mathrm{Sub}_w(n) \ \text{ and } \ |\,\mathrm{Sub}_w(n)| = L\,, \\ &\text{(b)} \quad \mathrm{Num}_w(v) = 1 \ \text{ for all } \ v \in \mathrm{Sub}_w(n+1)\,, \\ &\text{(c)} \quad |\,\mathrm{Sub}_w(n+1)| = |w| = kL\,. \end{aligned} \tag{3.21}$$

Obviously we have:

(a) and (b) holds if and only if (a) and (c) holds.

Let $[w]$ be a cyclic sequence which fulfill the above properties. By (b) and (c) follows, that $[w]$ is a closed path in $\mathcal{B}_q(n)$of length kL. By (a) follows, that the path runs through every of its vertices exactly k times. Therefore $[w]$ is an Euler circuit for some k-regular subgraph of $\mathcal{B}_q(n)$.

We call a cyclic sequence which fulfill the properties (a), (b) and (c) a (k, L, n)*-regular sequence*. Obviously a (k,L,n)-regular sequence has length kL. From the above remarks follows:

Proposition 44

(i) There exists a (k, L, n)-regular sequence if and only if there exists a k-regular subgraph of $\mathcal{B}_q(n)$ with L vertices.

(ii) If Λ is a k-regular subgraph of $\mathcal{B}_q(n)$ with L vertices, then every Euler circuit $[w]$ of Λ is a (k, L, n)-regular sequence.

(iii) If $[w]$ is a (k, L, n)-regular sequence, then $[w]$ is an Euler circuit of some k-regular subgraph of $\mathcal{B}_q(n)$ with L vertices, where $Sub_w(n)$ is the vertex set and $Sub_w(n+1)$ the edge set of the subgraph.

Lemma 45 *Let $w \in \mathcal{A}^+$ and let $[w]$ be a (k, L, n)-regular cyclic sequence. Then*

$$Num_w(u) \geq k^{n+1-l} \quad \forall\, u \in \ Sub_w(l)\,,\ 1 \leq l \leq n+1\,,$$

i.e. every letter of w occurs at least k^n times in w.

Proof: Let $w = w_0 \dots w_{L \cdot k}$ with $w_0, \dots, w_{L \cdot k-1} \in \mathcal{A}$. We show the lemma by induction on l:

By definition (3.21) (a) and (b) of (k, L, n)-regular sequences, we obtain that the lemma holds for $l \in \{n, n+1\}$.

Let $0 < l < n$ and let us assume that the lemma holds for $l+1$:

$$\mathrm{Num}_w(v) \geq k^{n+1-(l+1)} = k^{n-l} \text{ for all } v \in \mathrm{Sub}_w(l+1) . \tag{3.22}$$

We have to show that the lemma holds for l as well. If $u \in \mathrm{Sub}_w(l)$, then there is $0 \leq i < k \cdot L$ such that $u = w_i \dots w_{i+l-1}$. It follows:

$$uw_{i+l} \dots w_{i+n-1} \in \mathrm{Sub}_w(n) .$$

By the property (3.21) (a) of (k, L, n)-regular sequences follows:

$$\mathrm{Num}_w(uw_{i+l} \dots w_{i+n-1}) = k .$$

This shows, that there exists a (unique) set $A \subseteq \mathcal{A}$ with

$$|A| = k \text{ and } Auw_{i+l} \dots w_{i+n-1} \subseteq \mathrm{Sub}_w(n+1) ,$$

i.e. we obtain

$$Au \subseteq \mathrm{Sub}_w(l+1) .$$

By induction hypothesis (3.22)) follows $\mathrm{Num}_w(au) \geq k^{n-l}$ for all $a \in A$.

Thus we obtain:

$$\mathrm{Num}_w(u) \geq \sum_{a \in A} \mathrm{Num}_w(au) \geq |A| \cdot k^{n-l} = k^{n+1-l} .$$

The first inequality holds, because $\mathrm{Num}_w(u)$ is the number of occurrence of u as a subword in $[w]$ and therefore $\mathrm{Num}_w(u) = \sum_{a \in \mathcal{A}} \mathrm{Num}_w(au)$.

This shows that the lemma holds for all l with $0 < l \leq n+1$. The set of letters in w is given by $\mathrm{Sub}_w(1) \subseteq \mathcal{A}$. Since $\mathrm{Num}_w(a) \geq k^{n+1-1} = k^n$, every letter in $\mathrm{Sub}_w(1)$ occurs in w at least k^n times. **q.e.d**

We obtain with the one-to-one correspondence in Proposition 44 between (k, L, n)-regular sequences and k-regular subgraphs of $\mathcal{B}_q(n)$ with L vertices:

Theorem 14 *Let $n \in \mathbb{N}$ and $1 \leq k < q^n$. For every $L \in \mathbb{N}$ with $L < k^n$ or $k^n < L < k^n + k^{n-1}$ there does not exist a k-regular subgraph in $\mathcal{B}_q(n)$ with L vertices. Indeed there exists a connected k-regular subgraph in $\mathcal{B}_q(n)$ with k^n vertices as well as there exist one with $k^n - k^{n-1}$ vertices.*

Proof: We show first the second part of the lemma. Let $k \geq 2$. Obviously $\mathcal{B}_k(n)$ is a connected k-regular subgraph of $\mathcal{B}_q(n)$ with k^n vertices. ¿From Proposition 43 follows, that there exists a connected k-regular subgraph in $\mathcal{B}_q(n)$ with $k^n + k^{n-1}$ vertices if $k \geq 2$. If $k = 1$, then from Lempel's Theorem 39 follows that there exists a cycle of length $1 = 1^n$ and a cycle of length $2 = 1^n + 1^{n-1}$.This shows the second part of the lemma.

We show that the first part of the lemma holds for connected k-regular graphs. Let $L < k^n + k^{n-1}$ and Λ be a connected k-regular subgraph in $\mathcal{B}_q(n)$ with L vertices. By Proposition 44 follows that there exists a (k, L, n)-regular sequence $[w]$ for some $w \in \mathcal{A}^+$ with $|w| = k \cdot L < k^{n+1} + k^{n-1}$, such that $[w]$ is an Euler circuit of Λ. Let $\mathcal{A}' := \mathrm{Sub}_w(1)$ be the set of letters which occurs in w. It follows that Λ is also a connected k-regular subgraph of $\mathcal{B}_{\mathcal{A}'}(n)$ with L vertices. Since $\mathcal{B}_{|\mathcal{A}'|}(n) \cong \mathcal{B}_{\mathcal{A}'}(n)$, there exists a connected k-regular subgraph $\tilde{\Lambda}$ with L vertices in $\mathcal{B}_{|\mathcal{A}'|}(n)$, where $\tilde{\Lambda} \cong \Lambda$. With Lemma 45 follows, that every letter in w occurs at least k^n times in w. Thus:

$$\begin{aligned} & k \cdot L = |w| = \sum_{a \in \mathcal{A}'} \mathrm{Num}_w(a) \geq |\mathcal{A}'| \cdot k^n \\ \Rightarrow \quad & \tfrac{L}{k^{n-1}} = \tfrac{|w|}{k^n} \geq |\mathcal{A}'| \,. \end{aligned} \tag{3.23}$$

If $|\mathcal{A}'| < k$, then there does not exist a k-regular subgraph in $\mathcal{B}_{|\mathcal{A}'|}(n)$. Therefore from (3.23) follows, that $L \geq k^n$. Thus let $|\mathcal{A}'| \geq k$. Since $L < k^n + k^{n-1}$ it follows by (3.23), that $|\mathcal{A}'| = k$. Further on $\mathcal{B}_k(n)$ is the only k-regular subgraph in $\mathcal{B}_k(n)$. We conclude $\Lambda \cong L\tilde{a}mbda = \mathcal{B}_k(n)$ and $L = k^n$. This shows that there does not exist a connected k-regular subgraph of $\mathcal{B}_q(n)$ with L vertices, if $1 \leq L < k^n$ or $k^n < L < k^n + k^{n-1}$. Therefore the lemma holds for connected k-regular subgraphs of $\mathcal{B}_q(n)$. Furthermore we obtain that every connected k-regular subgraph of $\mathcal{B}_q(n)$ with k^n vertices is isomorphically to $\mathcal{B}_k(n)$.

Let us assume that there exist a unconnected k-regular subgraph Λ of $\mathcal{B}_q(n)$ with $L < k^n + k^{n-1}$ vertices. Let Λ_1, Λ_2 be two connectivity components of Λ. Then Λ_1, Λ_2 are vertex disjoint connected k-regular subgraphs in $\mathcal{B}_q(n)$ with less than $k^n + k^{n-1}$ vertices. It follows that Λ_1, Λ_2 are isomorphically to $\mathcal{B}_k(n)$. Therefore each of them has k^n vertices. It follows, that $2k^n \leq L$. This is a contradiction, because $k^n + k^{n-1} \leq 2k^n$. Therefore the lemma holds also for unconnected graphs. **q.e.d**

Chapter 4

Fix-free codes obtained from π-systems

In this chapter we will proof a generalization of a theorem of Yekhanin [8](2001), which shows that the $\frac{3}{4}$-conjecture holds for binary codes if the Kraftsum of the first level which occurs in the code together with it neighboring level is bigger than $\frac{1}{2}$. To show this, Yekhanin claimed two lemmas which imply the theorem. However in [8] no proof was given for the lemmas and due to my knowledge, no proof was published in other papers. The theorem and the sketch of the proof given in [8] is the following:

Theorem 15 (Yekhanin) *Let $|\mathcal{A}|$ =2 and $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers with $\sum\limits_{l=1}^{\infty} \alpha_l \left(\frac{1}{2}\right)^l \leq \frac{3}{4}$. If there exists an $n \in \mathbb{N}$ such that $\alpha_1 = \ldots = \alpha_{n-1} = 0$ and $\frac{\alpha_n}{2^n} + \frac{\alpha_{n+1}}{2^{n+1}} \geq \frac{1}{2}$, then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^*$ which fits to $(\alpha_l)_{l\in\mathbb{N}}$.*

For proving the theorem, Yekhanin introduced in [8] a special kind of fix-free codes, which he called π-systems:

Defenition 4 *Let $|\mathcal{A}| = 2$, we say $\mathcal{D} \subseteq \bigcup\limits_{l=1}^{n} \mathcal{A}^l$ is a π_2-system if $\mathcal{D}$ is fix-free with Kraftsum $\frac{1}{2}$ and*

$$|\Delta_S^n(\mathcal{D})| = |\Delta_P^n(\mathcal{D})| = |\mathcal{A}^{-1}\Delta_P^n(\mathcal{D})| = |\Delta_S^n(\mathcal{D})\mathcal{A}^{-1}| \tag{4.1}$$

Instead of (4.1) Yekanin defined in [8] π-systems with the following property:

$$|\mathcal{A}^n - \Delta_S^n(\mathcal{D})| = |\mathcal{A}^n - \Delta_P^n(\mathcal{D})| = |\mathcal{A}^{-1}(\mathcal{A}^n - \Delta_P^n(\mathcal{D})\,)| = |(\mathcal{A}^n - \Delta_S^n(\mathcal{D})\,)\mathcal{A}^{-1}| \tag{4.2}$$

If $\mathcal{D}$ is fix-free, from $S(\mathcal{D}) = \frac{1}{2}$ follows, that $|\Delta_P^n(\mathcal{D})| = |\Delta_S^n(\mathcal{D})| = 2^{n-1}$. Therefore the next proposition shows that for $\mathcal{D}$ fix-free with $S(\mathcal{D}) = \frac{1}{2}$, the properties (4.1) and (4.2) are equivalent.

Proposition 46 *Let $|\mathcal{A}| = q \geq 2$, $\mathcal{X} \subseteq \mathcal{A}^n$ and $\mathcal{X}^c := \mathcal{A}^n - \mathcal{X}$ then we have:*

$$\begin{array}{rl} & |\mathcal{X}| = |\mathcal{A}^{-1}\mathcal{X}| \geq q^{n-1} \\ \Leftrightarrow & |\mathcal{X}| = |\mathcal{A}^{-1}\mathcal{X}| = q^{n-1} \\ \Rightarrow & |\mathcal{A}^{-1}\mathcal{X}^c| = q^{n-1} \text{ and } |\mathcal{X}^c| = (q-1)q^{n-1} \end{array}$$

$$\begin{array}{rl} & |\mathcal{X}| = |\mathcal{X}\mathcal{A}^{-1}| \geq q^{n-1} \\ \Leftrightarrow & |\mathcal{X}| = |\mathcal{X}\mathcal{A}^{-1}| = q^{n-1} \\ \Rightarrow & |\mathcal{X}^c\mathcal{A}^{-1}| = q^{n-1} \text{ and } |\mathcal{X}^c| = (q-1)q^{n-1} \end{array}$$

Proof: From $\mathcal{X} \subseteq \mathcal{A}^n$ follows $|\mathcal{A}^{-1}\mathcal{X}| \leq |\mathcal{A}^{n-1}| = q^{n-1}$ and $|\mathcal{X}\mathcal{A}^{-1}| \leq q^{n-1}$. Obviously the equivalents in the proposition holds. Let $|\mathcal{X}| = |\mathcal{A}^{-1}\mathcal{X}| = q^{n-1}$. Then there exists for every $w \in \mathcal{A}^{n-1}$ a letter $a_w \in \mathcal{A}$ with $a_w w \in \mathcal{X}$ and a_w is unique because of the first equality. Therefore $\mathcal{X}^c$ is the (disjoint) union of the sets $(\mathcal{A} - \{a_w\})w$ with $w \in \mathcal{A}^{n-1}$. This shows $|\mathcal{A}^{-1}\mathcal{X}^c| = q^{n-1}$ and $|\mathcal{X}^c| = (q-1)q^{n-1}$ follows directly from $|\mathcal{X}| = q^{n-1}$. The second part of the proof of the proposition follows same steps. **q.e.d**

Theorem 15 follows from the two lemmas below:

Lemma 47 *Let $|\mathcal{A}|$ =2 and $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers with $\sum\limits_{l=1}^{\infty} \alpha_l \left(\frac{1}{2}\right)^l \leq \frac{3}{4}$. If there exists an $n \in \mathbb{N}$ and a π_2-system $\mathcal{D}$ such that $|\mathcal{A}^l \cap \mathcal{D}| = \alpha_l$ for all $1 \leq l \leq n$ and $|\mathcal{A}^{n+1} \cap \mathcal{D}| \leq \alpha_{n+1}$, then there exists fix-free extension $\mathcal{C}$ of $\mathcal{D}$ which fits $(\alpha_l)_{l\in\mathbb{N}}$*

Furthermore in the lemma above the codewords in $(\mathcal{C} - \mathcal{D})$ can chosen arbitrary by induction on the codeword lengths.

Lemma 48 *Let $n \in \mathbb{N}$, $\beta_1 = \ldots = \beta_{n-1} = 0$ and $\beta_n, \beta_{n+1} \in \mathbb{N}$ such that $\frac{\beta_n}{2^n} + \frac{\beta_{n+1}}{2^{n+1}} = \frac{1}{2}$, then there exists a π_2-system $\mathcal{D} \subseteq \mathcal{A}^{n+1}$ with $|\mathcal{A}^l \cap \mathcal{D}| = \beta_l$ for $1 \leq l \leq n+1$.*

In the next two sections we prove a generalization of the theorem for arbitrary finite alphabets. Therefore we give in the next section a more general definition of π-systems and a generalization of Lemma 47. In the second section of this chapter we show that there is a one-to-one correspondence between two level π-systems $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ and regular subgraphs in $\mathcal{B}_q(n-1)$, whereas the edges[1] of the corresponding regular subgraph are the codewords in $\mathcal{D}$ of length n. Especially for $|\mathcal{A}| = 2$ every cycle in $\mathcal{B}_2(n-1)$ is a 1-regular subgraph and as it was shown in the previous chapter, for every $1 \leq \beta_n \leq 2^{n-1}$ there exists

[1] Like in Chapter 3 we label the edges of $\mathcal{B}_q(n-1)$ with words of length n.

a β_n length cycle in $\mathcal{B}_2(n-1)$. With the one-to-one correspondence between regular subgraphs and two level π-systems of the form $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ we get Lemma 48. Finally we show in the second section of this chapter a generalization of Theorem 15 for arbitrary finite alphabets. However, because of the one-to-one correspondence between regular subgraphs and π-systems, in the general form of the theorem occurs a additional condition. This condition is the existence of regular subgraphs in $\mathcal{B}_q(n-1)$ with certain numbers of vertices.

4.1 Extensions of π-systems

In this section we give a generalization of Lemma 47 and introduce π-systems for arbitrary finite alphabets. For this we need some remarks about sets $\mathcal{X} \subseteq \mathcal{A}^n$ with the property $|\mathcal{A}^{-1}\mathcal{X}| = |\mathcal{X}|$ or $|\mathcal{X}\mathcal{A}^{-1}| = |\mathcal{X}|$.

Proposition 49 *Let* $|\mathcal{A}| = q \geq 2$.

(i) Let $\mathcal{X} \subseteq \mathcal{A}^n$ *then*
$|\mathcal{X}| = |\mathcal{A}^{-1}\mathcal{X}| \Leftrightarrow$ *if* $w_1 \dots w_n \in \mathcal{X}, a \in \mathcal{A} - \{w_1\}$ *then* $aw_2 \dots w_n \notin \mathcal{X}$
$|\mathcal{X}| = |\mathcal{X}\mathcal{A}^{-1}| \Leftrightarrow$ *if* $w_1 \dots w_n \in \mathcal{X}, a \in \mathcal{A} - \{w_n\}$ *then* $w_1 \dots w_{n-1}a \notin \mathcal{X}$

(ii) Let $\mathcal{X} \subseteq \mathcal{A}^n$ *then*
$|\mathcal{X}| = |\mathcal{A}^{-1}\mathcal{X}| \Leftrightarrow |\mathcal{A}^{-1}\mathcal{X}\mathcal{A}^l| = |\mathcal{X}\mathcal{A}^l| \; \forall \, l \in \mathbb{N}$
$|\mathcal{X}| = |\mathcal{X}\mathcal{A}^{-1}| \Leftrightarrow |\mathcal{A}^l\mathcal{X}\mathcal{A}^{-1}| = |\mathcal{A}^l\mathcal{X}| \; \forall \, l \in \mathbb{N}$

(iii) Let $\mathcal{X} \subseteq \bigcup_{l=1}^{n} \mathcal{A}^l$ *and* $\mathcal{X}_l := \mathcal{X} \cap \mathcal{A}^l$ *then*
$|\mathcal{A}^{-1}\mathcal{X}| = |\mathcal{X}| \Leftrightarrow |\mathcal{A}^{-1}\mathcal{X}_l| = |\mathcal{X}_l| \; \forall \, l \in \mathbb{N}$
$|\mathcal{X}\mathcal{A}^{-1}| = |\mathcal{X}| \Leftrightarrow |\mathcal{X}_l\mathcal{A}^{-1}| = |\mathcal{X}_l| \; \forall \, l \in \mathbb{N}$

(iv) Let $\mathcal{X} \subseteq \bigcup_{l=1}^{n} \mathcal{A}^l$ *then we have for every* $N \geq n$
$|\Delta_P^n(\mathcal{X})| = |\mathcal{A}^{-1}\Delta_P^n(\mathcal{X})| \Leftrightarrow |\Delta_P^N(\mathcal{X})| = |\mathcal{A}^{-1}\Delta_P^N(\mathcal{X})|$
$|\Delta_S^n(\mathcal{X})| = |\Delta_S^n(\mathcal{X})\mathcal{A}^{-1}| \Leftrightarrow |\Delta_S^N(\mathcal{X})| = |\Delta_S^N(\mathcal{X})\mathcal{A}^{-1}|$

Proof: (i) is obvious. For (ii) we have

$$\begin{array}{rcl} |\mathcal{X}| & = & |\mathcal{A}^{-1}\mathcal{X}| \\ \Leftrightarrow \quad |\mathcal{X}| \cdot |\mathcal{A}^l| & = & |\mathcal{A}^{-1}\mathcal{X}| \cdot |\mathcal{A}^l| \\ \Leftrightarrow \quad \left|\mathcal{X}\mathcal{A}^l\right| & = & \left|\mathcal{A}^{-1}\mathcal{X}\mathcal{A}^l\right| \end{array}$$

This shows the first part of (ii), the second part (ii) follows the same steps.
For the first part of (iii) take in account that:

$$|\mathcal{X}| = |\mathcal{A}^{-1}\mathcal{X}| \Leftrightarrow \sum_{l=1}^{n} |\mathcal{X}_l| = \sum_{l=1}^{n} |\mathcal{A}^{-1}\mathcal{X}_l| \, .$$

Since the terms in the sums are nonnegative and $|\mathcal{X}_l| \geq |\mathcal{A}^{-1}\mathcal{X}_l|$ for all l with $1 \leq l \leq n$, the second equation holds only if $|\mathcal{X}_l| = |\mathcal{A}^{-1}\mathcal{X}_l|$ for all l with $1 \leq l \leq n$. In the same way follows the second part of (iii).
(iv) follows now from (ii), because

$$\Delta_P^N(\mathcal{X}) = \Delta_P^n(\mathcal{X})\mathcal{A}^{N-n}\,, \quad \mathcal{A}^{-1}\Delta_P^N(\mathcal{X}) = \mathcal{A}^{-1}\Delta_P^n(\mathcal{X})\mathcal{A}^{N-n}$$

$$\Delta_S^N(\mathcal{X}) = \mathcal{A}^{N-n}\Delta_S^n(\mathcal{X})\,, \quad \Delta_S^N(\mathcal{X})\mathcal{A}^{-1} = \mathcal{A}^{N-n}\Delta_S^n(\mathcal{X})\mathcal{A}^{-1}$$

q.e.d

Lemma 50 *Let* $\mathcal{X} \subseteq \bigcup_{l=1}^{n} \mathcal{A}^l$, $N \geq n$ *and* $\mathcal{X}_l := \mathcal{X} \cap \mathcal{A}^l$ *then:*

(i)

$$\begin{aligned} & \left|\Delta_P^N(\mathcal{X})\right| = \left|\mathcal{A}^{-1}\Delta_P^N(\mathcal{X})\right| \quad \text{and } \mathcal{X} \text{ is prefix-free} \\ \Leftrightarrow\ & \mathcal{A}^{-1}\mathcal{X} \quad \text{is prefix-free and } |\mathcal{A}^{-1}\mathcal{X}_l| = |\mathcal{X}_l| \quad \text{for all } 1 \leq l \leq n, \\ \Leftrightarrow\ & \mathcal{A}^{-1}\mathcal{X} \quad \text{is prefix-free and } |\mathcal{A}^{-1}\mathcal{X}| = |\mathcal{X}|\,. \end{aligned}$$

(ii)

$$\begin{aligned} & \left|\Delta_S^N(\mathcal{X})\right| = \left|\mathcal{A}^{-1}\Delta_S^N(\mathcal{X})\right| \quad \text{and } \mathcal{X} \text{ is suffix-free} \\ \Leftrightarrow\ & \mathcal{X}\mathcal{A}^{-1} \quad \text{is suffix-free and } |\mathcal{X}_l\mathcal{A}^{-1}| = |\mathcal{X}_l| \quad \text{for all } 1 \leq l \leq n, \\ \Leftrightarrow\ & \mathcal{X}\mathcal{A}^{-1} \quad \text{is suffix-free and } |\mathcal{X}\mathcal{A}^{-1}| = |\mathcal{X}|\,. \end{aligned}$$

Proof: Let $\left|\Delta_P^N(\mathcal{X})\right| = \left|\mathcal{A}^{-1}\Delta_P^N(\mathcal{X})\right|$ and $\mathcal{X}$ be prefix-free. If we assume that $\mathcal{A}^{-1}\mathcal{X}$ is not prefix-free, then there exists $u, v \in \mathcal{A}^{-1}\mathcal{X}$ and $a, b \in \mathcal{A}$ such that $u = vu'$ for some $u' \in \mathcal{A}^+$ and $au, bv \in \mathcal{X}$. Then $a \neq b$, because $\mathcal{X}$ is prefix-free. It follows:

$$au\mathcal{A}^{N-|u|-1} \subseteq \Delta_P^N(\mathcal{X}) \;\text{ and }\; bvu'\mathcal{A}^{N-|vu'|-1} = bu\mathcal{A}^{N-|u|-1} \subseteq \Delta_P^N(\mathcal{X})$$

By Proposition 49 (i) and $a \neq b$ follows $\left|\Delta_P^N(\mathcal{X})\right| > \left|\mathcal{A}^{-1}\Delta_P^N(\mathcal{X})\right|$. This is a contradiction. Therefore $\mathcal{A}^{-1}\mathcal{X}$ is prefix-free.
If we take in account that also $\mathcal{X}$ is prefix-free, we obtain:

$$\left|\Delta_P^N(\mathcal{X})\right| = \left|\mathcal{A}^{-1}\Delta_P^N(\mathcal{X})\right| \Leftrightarrow \sum_{l=1}^{n} |\mathcal{X}_l| \cdot |\mathcal{A}^{N-l}| = \sum_{l=1}^{n} |\mathcal{A}^{-1}\mathcal{X}_l| \cdot |\mathcal{A}^{N-l}|$$

While $0 \geq |\mathcal{A}^{-1}\mathcal{X}_l| \cdot |\mathcal{A}^{N-l}| < |\mathcal{X}_l| \cdot |\mathcal{A}^{N-l}|$ for all l with $1 \leq l \leq n$, the second equation holds only if $|\mathcal{X}_l| = |\mathcal{A}^{-1}\mathcal{X}_l|$ holds for all l with $1 \leq l \leq n$. This shows that for the prefix-free sets $\mathcal{X}$ and $\mathcal{A}^{-1}\mathcal{X}$ the following equivalence is true:

$$\left|\Delta_P^N(\mathcal{X})\right| = \left|\mathcal{A}^{-1}\Delta_P^N(\mathcal{X})\right| \Leftrightarrow |\mathcal{X}_l| = |\mathcal{A}^{-1}\mathcal{X}_l| \quad \forall\, 1 \leq l \leq n \tag{4.3}$$

Therefore we obtain $\mathcal{A}^{-1}\mathcal{X}$ is prefix-free and $|\mathcal{X}_l| = |\mathcal{A}^{-1}\mathcal{X}_l| \quad \forall\, 1 \leq l \leq n$.

Let us assume that $\mathcal{A}^{-1}\mathcal{X}$ is prefix-free and $|\mathcal{X}_l| = |\mathcal{A}^{-1}\mathcal{X}_l| \quad \forall\, 1 \leq l \leq n$. Then from the assumption that $\mathcal{A}^{-1}\mathcal{X}$ is prefix-free follows that $\mathcal{X}$ is prefix-free as well and by (4.3) we obtain, that also $\left|\Delta_P^N(\mathcal{X})\right| = \left|\mathcal{A}^{-1}\Delta_P^N(\mathcal{X})\right|$ holds. This shows the first equivalence of (i).

The second equivalence of (i) follows by Proposition 49 (iii). The proof for (ii) follows the same step as the proof of (i). **q.e.d**

Defenition 5

*Let $|\mathcal{A}| = q \geq 2$, $1 \leq k \leq q$ and $n \in \mathbb{N}$. We call a set $\mathcal{D} \subseteq \bigcup_{l=1}^{n} \mathcal{A}^l$ a $\pi_q(n;k)$-*system *if $\mathcal{D}$ is fix-free, and there exists a partition of $\mathcal{D}$ into k sets $\mathcal{D}_1, \ldots, \mathcal{D}_k$ for which the following three equivalent properties holds.*

(1): For all $1 \leq i \leq k$ holds:

$$\begin{aligned} q^{n-1} &= |\Delta_P^n(\mathcal{D}_i)| = |\mathcal{A}^{-1}\Delta_P^n(\mathcal{D}_i)| \\ &= |\Delta_S^n(\mathcal{D}_i)| = |\Delta_S^n(\mathcal{D}_i)\mathcal{A}^{-1}| \end{aligned}$$

(2): $S(\mathcal{D}) = \frac{k}{q}$ and for all i with $1 \leq i \leq k$ holds:

$$|\Delta_P^n(\mathcal{D}_i)| = |\mathcal{A}^{-1}\Delta_P^n(\mathcal{D}_i)| \quad \textit{and} \quad |\Delta_S^n(\mathcal{D}_i)| = |\Delta_S^n(\mathcal{D}_i)\mathcal{A}^{-1}|$$

(3): For all $1 \leq i \leq k$ the set $\mathcal{A}^{-1}\mathcal{D}_i$ is maximal prefix-free, $\mathcal{D}_i\mathcal{A}^{-1}$ is maximal suffix-free and $|\mathcal{A}^{-1}\mathcal{D}_i| = |\mathcal{D}_i\mathcal{A}^{-1}| = |\mathcal{D}_i|$.

The sets $\mathcal{D}_1, \ldots, \mathcal{D}_k$ are called a π-partition of $\mathcal{D}$

For $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$ we call a $\pi_q(n;k)$-system $\mathcal{D}$ a $\pi_q(\alpha_1, \ldots, \alpha_n;k)$-system if $\left|\mathcal{D} \cap \mathcal{A}^l\right| = \alpha_l$ for all $1 \leq l \leq n$.

We show that (1)-(3) in the definition are all equivalent. Therefore let $\mathcal{D} \subseteq \bigcup_{l=1}^{n} \mathcal{A}^l$ be a fix-free code and $\mathcal{D}_1 \ldots, \mathcal{D}_k$ be a partition of $\mathcal{D}$.

(1) $\Rightarrow$ (3)**:**
Let $\mathcal{D}_1 \ldots, \mathcal{D}_k$ such that (1) holds. Since $\mathcal{D}$ is fix-free, all $\mathcal{D}_i$ are fix-free. With Lemma 50 and property (1) follows that $\mathcal{A}^{-1}\mathcal{D}_i$ is prefix-free and $\mathcal{D}_i\mathcal{A}^{-1}$ is suffix-free.

We obtain for all $1 \leq i \leq k$:

$$\begin{aligned}
q^{n-1} &= |\mathcal{A}^{-1}\Delta_P^n(\mathcal{D}_i)| = |\mathcal{A}^{-1}\bigcup_{l=1}^{n}(\mathcal{D}_i \cap \mathcal{A}^l)\mathcal{A}^{n-l}| = |\bigcup_{l=1}^{n}\mathcal{A}^{-1}(\mathcal{D}_i \cap \mathcal{A}^l)\mathcal{A}^{n-l}| \\
&\overset{\substack{\mathcal{A}^{-1}\mathcal{D} \text{ is} \\ \text{prefix-free}}}{=} \sum_{l=1}^{n}|\mathcal{A}^{-1}(\mathcal{D}_i \cap \mathcal{A}^l)\mathcal{A}^{n-l}| = \sum_{l=1}^{n}|\mathcal{A}^{-1}D_i \cap \mathcal{A}^{l-1}| \cdot q^{n-l} \\
&= \sum_{l=0}^{n-1}|\mathcal{A}^{-1}\mathcal{D}_i \cap \mathcal{A}^l| \cdot q^{n-l-1}\,.
\end{aligned}$$

It follows:

$$S(\mathcal{A}^{-1}\mathcal{D}_i) = \sum_{l=0}^{n-1}|\mathcal{A}^{-1}\mathcal{D}_i \cap \mathcal{A}^l| \cdot q^{-l} = 1\,.$$

This shows that $\mathcal{A}^{-1}\mathcal{D}_i$ is maximal prefix-free. In the same way follows that $\mathcal{D}_i\mathcal{A}^{-1}$ is maximal suffix-free. Furthermore we obtain $|\mathcal{A}^{-1}\mathcal{D}_i| = |\mathcal{D}_i\mathcal{A}^{-1}| = |\mathcal{D}_i|$ from Lemma 50. Therefore (3) holds for the sets $\mathcal{D}_1, \ldots, \mathcal{D}_k$.

(3) $\Rightarrow$ (2)**:**
Let $\mathcal{D}_1 \ldots, \mathcal{D}_k$ be such that (3) holds. Then from Lemma 50 follows

$$|\Delta_P^n(\mathcal{D}_i)| = |\mathcal{A}^{-1}\Delta_P^n(\mathcal{D}_i)| \ \text{ and } \ |\Delta_S^n(\mathcal{D}_i)| = |\Delta_S^n(\mathcal{D}_i)\mathcal{A}^{-1}| \ \text{ for all } \ 1 \leq i \leq k\,.$$

Therefore we have to show: $S(\mathcal{D}) = \frac{k}{q}$. Because of Lemma 50 we have $|\mathcal{A}^{-1}(\mathcal{D}_i \cap \mathcal{A}^l)| = |\mathcal{D}_i \cap \mathcal{A}^l|$ for all $1 \leq i \leq k$ and $1 \leq l \leq n$. Since $\mathcal{A}^{-1}\mathcal{D}_i$ is maximal prefix-free ($S(\mathcal{A}^{-1}\mathcal{D}_i) = 1$), we obtain :

$$\begin{aligned}
1 &= S(\mathcal{A}^{-1}\mathcal{D}_i) = \sum_{l=0}^{n-1}|\mathcal{A}^{-1}\mathcal{D}_i \cap \mathcal{A}^l| \cdot q^{-l} = \sum_{l=0}^{n-1}|\mathcal{A}^{-1}(\mathcal{D}_i \cap \mathcal{A}^{l+1})| \cdot q^{-l} \\
&= \sum_{l=1}^{n}|\mathcal{A}^{-1}(\mathcal{D}_i \cap \mathcal{A}^l)| \cdot q^{-l+1} = q \cdot \sum_{l=1}^{n}|\mathcal{D}_i \cap \mathcal{A}^l| \cdot q^{-l} = q \cdot S(\mathcal{D}_i)
\end{aligned}$$

Therefore $S(\mathcal{D}_i) = \frac{1}{q}$ for all $1 \leq i \leq k$. Because $\mathcal{D}_1, \ldots, \mathcal{D}_k$ is a partition of $\mathcal{D}$ it follows:

$$S(\mathcal{D}) = S(\mathcal{D}_1) + \ldots + S(\mathcal{D}_k) = \frac{k}{q}$$

This shows that (2) holds for $\mathcal{D}_1, \ldots, \mathcal{D}_k$.

$(2) \Rightarrow (1)$**:**
Let $\mathcal{D}_1 \ldots, \mathcal{D}_k$ be such that (2) holds. We have to show that $|\Delta_P^n(\mathcal{D}_i)| = |\Delta_S^n(\mathcal{D}_i)| = q^{n-1}$ holds for all $1 \leq i \leq k$. Since $\mathcal{A}^{-1}\Delta_P^n(\mathcal{D}_i) \subseteq \mathcal{A}^{n-1}$ from (2) follows :

$$|\Delta_P^n(\mathcal{D}_i)| = \left|\Delta_P^n \mathcal{A}^{-1}(\mathcal{D}_i)\right| \leq q^{n-1} \,\forall\, 1 \leq i \leq k\,. \tag{4.4}$$

While the Kraftsum of $\mathcal{D}$ is equal to $\frac{k}{q}$, we get:

$$\begin{aligned} k \cdot q^{n-1} &= \sum_{l=1}^{n} \left|\mathcal{D} \cap \mathcal{A}^l\right| \cdot q^{n-l} \\ (\mathcal{D} \text{ is prefix-free}) &= |\Delta_P^n(\mathcal{D})| \\ (\text{ the } \mathcal{D}_i\text{'s are a partition of } \mathcal{D}\ ) &= \sum_{i=1}^{k} |\Delta_P^n(\mathcal{D}_i)|\ . \end{aligned}$$

¿From the last equality and (4.4) follows:

$$|\Delta_P^n(\mathcal{D}_i)| = q^{n-1}\ mboxforall\ 1 \leq i \leq k\,.$$

Similar arguments show that also $|\Delta_S^n(\mathcal{D}_i)| = q^{n-1}$ holds for all $1 \leq i \leq k$. Thus $\mathcal{D}_1, \ldots, \mathcal{D}_k$ have property (1) as well.

It follows, that for a fix-free code $\mathcal{D}$ with partition $\mathcal{D}_1, \ldots, D_k$ (1), (2) and (3) in the Definition 5 are all equivalent. Furthermore we get from (2), that the Definition 5 of $\pi_q(n;1)$-systems coincides for $q = 2$ with the first definition of π_2-systems.

Lemma 51 *Let* $|\mathcal{A}| = q < \infty$

(i) Let $\mathcal{Y} \subseteq \mathcal{A}^n, \mathcal{X} \subseteq \mathcal{A}^{n-1}$ *then we have:*

a) If $|\mathcal{Y}\mathcal{A}^{-1}| = |\mathcal{Y}| = q^{n-1}$ *then* $|\mathcal{X}\mathcal{A} \cap \mathcal{Y}| = |\mathcal{X}|$.

b) If $|\mathcal{A}^{-1}\mathcal{Y}| = |\mathcal{Y}| = q^{n-1}$ *then* $|\mathcal{A}\mathcal{X} \cap \mathcal{Y}| = |\mathcal{X}|$.

(ii) Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}^n$ *for some* $n \geq 1$ *then we have:*
$|\mathcal{A}^{-1}\mathcal{X}| = |\mathcal{X}|$ *and* $|\mathcal{Y}\mathcal{A}^{-1}| = |\mathcal{Y}| \Rightarrow |\mathcal{X}\mathcal{A} \cap \mathcal{A}\mathcal{Y}| \geq |\mathcal{X}| + |\mathcal{Y}| - q^{n-1}$

Proof:

(i): Let $\mathcal{A} := \{a_1, \ldots, a_q\}$. We prove only part a), because the proof of part b) is analogously. Therefore let $\mathcal{Y} \subseteq \mathcal{A}^n, \mathcal{X} \subseteq \mathcal{A}^{n-1}$ and $|\mathcal{Y}\mathcal{A}^{-1}| = |\mathcal{Y}| = q^{n-1}$. Let $\mathcal{Y}_l := \mathcal{Y}a_l^{-1}$, then $\mathcal{Y}$ is the disjoint union of $\mathcal{Y}_1 a_1, \ldots, \mathcal{Y}_q a_q$. We claim, that $\mathcal{Y}_1, \ldots \mathcal{Y}_q$ are pairwise disjoint. Assume that $\mathcal{Y}_l \cap \mathcal{Y}_k \neq \emptyset$ for some $l \neq k$, then there exists some $w \in \mathcal{A}^{n-1}$, such that $wa_l, wa_k \in \mathcal{Y}$. This is a contradiction, because $|\mathcal{Y}\mathcal{A}^{-1}| = |\mathcal{Y}|$. Therefore $\mathcal{Y}\mathcal{A}^{-1}$ is the disjoint union of $\mathcal{Y}_1, \ldots, \mathcal{Y}_q$. Since $|\mathcal{Y}\mathcal{A}^{-1}| = q^{n-1} = |\mathcal{A}^{n-1}|$, the sets $\mathcal{Y}_1, \ldots, \mathcal{Y}_q$ are a partition of $\mathcal{A}^{n-1}$.
Thus we get:

$$|\mathcal{X}\mathcal{A} \cap \mathcal{Y}| = \sum_{l=1}^{q} |\mathcal{X}\mathcal{A} \cap \mathcal{Y}_l a_l| = \sum_{l=1}^{q} |\mathcal{X} \cap \mathcal{Y}_l| = |\mathcal{X} \cap \mathcal{A}^{n-1}| = |\mathcal{X}|\,.$$

(ii): By $|\mathcal{A}^{-1}\mathcal{X}| = |\mathcal{X}|$ and $|\mathcal{Y}\mathcal{A}^{-1}| = |\mathcal{Y}|$ we have:

$$\begin{aligned} q^{n-1} = |\mathcal{A}^{n-1}| &\geq |\mathcal{A}^{-1}\mathcal{X} \cup \mathcal{Y}\mathcal{A}^{-1}| \\ &= |\mathcal{A}^{-1}\mathcal{X}| + |\mathcal{Y}\mathcal{A}^{-1}| - |\mathcal{A}^{-1}\mathcal{X} \cap \mathcal{Y}\mathcal{A}^{-1}| \\ &= |\mathcal{X}| + |\mathcal{Y}| - |\mathcal{A}^{-1}\mathcal{X} \cap \mathcal{Y}\mathcal{A}^{-1}| \end{aligned}$$

and therefore we obtain:

$$|\mathcal{A}^{-1}\mathcal{X} \cap \mathcal{Y}\mathcal{A}^{-1}| \geq |\mathcal{X}| + |\mathcal{Y}| - q^{n-1}\,. \tag{4.5}$$

For every $w \in \mathcal{A}^{-1}\mathcal{X} \cap \mathcal{Y}\mathcal{A}^{-1}$ there exist $a, b \in \mathcal{A}$ with $aw \in \mathcal{X}$ and $wb \in \mathcal{Y}$. It follows that $awb \in \mathcal{X}\mathcal{A} \cap \mathcal{A}\mathcal{Y}$. Since $|\mathcal{A}^{-1}\mathcal{X}| = |\mathcal{X}|$ and $|\mathcal{Y}\mathcal{A}^{-1}| = |\mathcal{Y}|$, the letters a, b are unique. Vice versa, for $v \in \mathcal{X}\mathcal{A} \cap \mathcal{A}\mathcal{Y}$ there are $a, b \in \mathcal{A}$ and $w \in \mathcal{A}^{n-1}$ such that $aw \in \mathcal{X}$ and $wb \in \mathcal{Y}$. It follows that $w \in \mathcal{A}^{-1}\mathcal{X} \cap \mathcal{Y}\mathcal{A}^{-1}$. This gives us a one-to-one map from $\mathcal{A}^{-1}\mathcal{X} \cap \mathcal{Y}\mathcal{A}^{-1}$ onto $\mathcal{X}\mathcal{A} \cap \mathcal{A}\mathcal{Y}$, and therefore

$$|\mathcal{A}^{-1}\mathcal{X} \cap \mathcal{Y}\mathcal{A}^{-1}| = |\mathcal{X}\mathcal{A} \cap \mathcal{A}\mathcal{Y}|\,.$$

Together with (4.5) follows: $|\mathcal{X}\mathcal{A} \cap \mathcal{A}\mathcal{Y}| \geq |\mathcal{X}| + |\mathcal{Y}| - q^{n-1}$. **q.e.d**

The following theorem shows that $\pi_q(n)$-systems can always be extended to a fix-free code, if the Kraftsum is smaller than or equal to $\frac{3}{4}$. It is a generalization of Lemma 48 for arbitrary finite alphabets .

Theorem 16 (π-system extension theorem) *For $|\mathcal{A}| = q \geq 2$ and $1 \leq k < q$ let*

$$\gamma_k := \begin{cases} \frac{1}{2} + \frac{k}{2q} & \text{for } 1 \leq k \leq \lfloor \frac{q}{2} \rfloor \\ \left(\frac{q-k}{q}\right)^2 + \frac{k}{q} & \text{for } \lfloor \frac{q}{2} \rfloor < k < q \end{cases}$$

Let $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers and let $n \in \mathbb{N}$, $1 \leq \beta \leq \alpha_n$ be such that:

$$\sum_{l\in\mathbb{N}}^{\infty} \alpha_l q^{-l} > \frac{k}{q} \quad \text{and} \quad \beta q^{-n} + \sum_{l=1}^{n-1} \alpha_l q^{-l} = \frac{k}{q}.$$

If $\sum\limits_{l=1}^{\infty} \alpha_l q^{-l} \leq \gamma_k$, then for every $\pi_q(\alpha_1, \ldots, \alpha_{n-1}, \beta; k)$-system there exists a fix-free extension which fits to $(\alpha_l)_{l\in\mathbb{N}}$.

Note that $\gamma_k > \frac{k}{q}$ for $1 \leq k \leq q$ and that there exist unique $\beta, n \in \mathbb{N}$ with the properties in the theorem.

Furthermore the proof of the theorem will show, that an extension $\mathcal{C}$ of a $\pi_q(\alpha_1, \ldots, \alpha_{n-1}\beta; k)$-system $\mathcal{D}$ can be constructed as follows:

1. add to $\mathcal{D}$ $(\alpha_n - \beta)$ arbitrary codewords of length n which are not in $\Delta_B^n(\mathcal{D})$ to obtain a fix-free $\mathcal{C}_0$.
2. For $m \in \mathbb{N}$ add to $\mathcal{C}_{m-1}$ α_{n+m} arbitrary codewords of length $(n+m)$ which are not in $\Delta_B^{n+m}(\mathcal{D}_{m-1})$ to obtain a fix-free $\mathcal{C}_m$.
3. Take the union of all $\mathcal{C}_m$'s to obtain the fix-free extension $\mathcal{C}$.

Proof: Let q and k as in the theorem. We claim:

$$\gamma_k = \min\left\{\frac{1}{2} + \frac{k}{2q} \quad , \ \left(\frac{q-k}{q}\right)^2 + \frac{k}{q}\right\} \quad \forall\, 1 \leq k < q\,. \tag{4.6}$$

Let $f(x) := \frac{q+x}{2q}$ and $g(x) := \left(\frac{q-x}{q}\right)^2 + \frac{x}{q}$, then:

$$f(0) = \frac{1}{2} < 1 = g(0) \quad \text{and} \quad f(x) = g(x) \Leftrightarrow x \in \{\frac{q}{2}, q\} \tag{4.7}$$

Since f and g are continuous functions, equation (4.6) follows from (4.7).

Let $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers with $\sum_{l=1}^{\infty} \alpha_l q^{-l} \le \frac{3}{4}$ and choose $\beta, n \in \mathbb{N}$ such that $\beta q^{-n} + \sum_{l=1}^{n-1} \alpha_l q^{-l} = \frac{k}{q}$ Let $\mathcal{D}$ be a $\pi_q(\alpha_1, \ldots, \alpha_{n-1}, \beta; k)$-system, with π-partition $\mathcal{D}_1, \ldots, \mathcal{D}_k$. We will show by a simple induction on the codeword length, that the construction of a fix-free code which fits $(\alpha_l)_{l\in\mathbb{N}}$ is possible in the way described above. Because of (1) in the definition of π-systems and Proposition 49 (iv) we get for all $m \in \mathbb{N}$ and $1 \le i \le k$:

$$\left|\mathcal{A}^{-1}\Delta_P^{n+m}(\mathcal{D}_i)\right| = \left|\Delta_P^{n+m}(\mathcal{D}_i)\right| = q^m \left|\Delta_P^n(\mathcal{D}_i)\right| = q^{n+m-1}, \tag{4.8}$$

$$\left|\Delta_S^{n+m}(\mathcal{D}_i)\mathcal{A}^{-1}\right| = \left|\Delta_S^{n+m}(\mathcal{D}_i)\right| = q^m \left|\Delta_S^n(\mathcal{D}_i)\right| = q^{n+m-1}. \tag{4.9}$$

Since $\mathcal{D}$ is fix-free and the disjoint union of the $\mathcal{D}_i$'s, it follows:

$$\left|\Delta_P^{n+m}(\mathcal{D})\right| = \left|\Delta_S^{n+m}(\mathcal{D})\right| = q^{n+m-1}k \text{ for all } m \in \mathbb{N}. \tag{4.10}$$

Case m=0:
We show, that the cardinality of the bifix-shadow of $\mathcal{D}$ on the n-th level is smaller than $|\mathcal{A}^n| - (\alpha_n - \beta) = q^n + \beta - \alpha_n$. Then we can add $(\alpha_n - \beta)$ codewords of length n to $\mathcal{D}$ and obtain a fix-free code $\mathcal{C}_0 \supseteq \mathcal{D}$ which fits to $(\alpha_1, \ldots, \alpha_n)$.

Let for $1 \le i \le k$

$$\begin{aligned} \mathcal{F}_i &:= \mathcal{D}_i \cap \mathcal{A}^n \quad , \quad \mathcal{F} := \bigcup_{i=1}^{k} \mathcal{F}_i = \mathcal{D} \cap \mathcal{A}^n, \\ \mathcal{E}_i &:= \mathcal{D}_i - \mathcal{F}_i \quad , \quad \mathcal{E} := \bigcup_{i=1}^{k} \mathcal{E}_i = \mathcal{D} - \mathcal{F} \end{aligned} ,$$

because the $\mathcal{D}_i$'s are pairwise disjoint and $\mathcal{D}$ is fix-free it follows that also $\mathcal{F}_1, \ldots \mathcal{F}_k$, $\mathcal{E}_1, \ldots \mathcal{E}_k$ are pairwise disjoint and fix-free. Furthermore we obtain
for all $1 \le i \le k$:

$$\Delta_P^n(\mathcal{F}_i) = \Delta_S^n(\mathcal{F}_i) = \mathcal{F}_i, \quad \Delta_P^n(\mathcal{F}) = \Delta_S^n(\mathcal{F}) = \mathcal{F},$$

$$\Delta_P^n(\mathcal{E}_i) = \Delta_P^{n-1}(\mathcal{E}_i)\mathcal{A} \text{ and } \Delta_S^n(\mathcal{E}_i) = \mathcal{A}\Delta_S^{n-1}(\mathcal{E}_i).$$

With Lemma 51 (i) and property (1) of the $\mathcal{D}_i$'s in the definition of π-systems follows for all $1 \leq i, j \leq k$:

$$|\Delta_P^n(\mathcal{E}_i) \cap \Delta_S^n(\mathcal{D}_j)| = \left|\Delta_P^{n-1}(\mathcal{E}_i)\mathcal{A} \cap \Delta_S^n(\mathcal{D}_j)\right| = \left|\Delta_P^{n-1}(\mathcal{E}_i)\right| , \tag{4.11}$$

$$|\Delta_S^n(\mathcal{E}_i) \cap \Delta_P^n(\mathcal{D}_j)| = \left|\mathcal{A}\Delta_S^{n-1}(\mathcal{E}_i) \cap \Delta_P^n(\mathcal{D}_j)\right| = \left|\Delta_S^{n-1}(\mathcal{E}_i)\right| . \tag{4.12}$$

The sets $\Delta_P^n(\mathcal{E}_1), \ldots, \Delta_P^n(\mathcal{E}_k)$ as well as the sets $\Delta_P^n(\mathcal{D}_1), \ldots, \Delta_P^n(\mathcal{D}_k)$ are pairwise disjoint, because $\mathcal{E}, \mathcal{D}$ are fix-free and both the $\mathcal{E}_i$'s and $\mathcal{D}_i$'s are pairwise disjoint. It follows:

$$\begin{array}{rcl}
|\Delta_P^n(\mathcal{E}) \cap \Delta_S^n(\mathcal{D})| & = & \left|\bigcup_{i=1}^{k} \Delta_P^n(\mathcal{E}_i) \cap \bigcup_{j=1}^{k} \Delta_P^n(\mathcal{D}_j)\right| \\
 & = & \left|\bigcup_{i,j=1}^{k} \left(\Delta_P^n(\mathcal{E}_i) \cap \Delta_S^n(\mathcal{D}_j)\right)\right| \\
 & = & \sum_{i,j=1}^{k} |\Delta_P^n(\mathcal{E}_i) \cap \Delta_S^n(\mathcal{D}_j)| \\
\text{with (4.11)} & = & \sum_{i,j=1}^{k} \left|\Delta_P^{n-1}(\mathcal{E}_i)\right| = k \cdot \sum_{i=1}^{k} \left|\Delta_P^{n-1}(\mathcal{E}_i)\right| \\
 & = & k \cdot \sum_{i=1}^{k} \frac{|\Delta_P^n(\mathcal{E}_i)|}{q} = \frac{k}{q} \cdot |\Delta_P^n(\mathcal{E})| .
\end{array}$$

Therefore we have:

$$|\Delta_P^n(\mathcal{E}) \cap \Delta_S^n(\mathcal{D})| = \frac{k}{q} \cdot |\Delta_P^n(\mathcal{E})| . \tag{4.13}$$

For the codewords of $\mathcal{D}$ which are contained in the prefix-shadow and in the suffix-shadow we obtain:

$$\begin{array}{rcl}
|\Delta_P^n(\mathcal{D}) \cap \Delta_S^n(\mathcal{D})| & = & |(\Delta_P^n(\mathcal{E}) \cap \Delta_S^n(\mathcal{D})) \cup (\mathcal{F} \cap \Delta_S^n(\mathcal{D}))| \\
{\scriptstyle \Delta_P^n(\mathcal{E}) \cap \mathcal{F} = \emptyset \text{ because} \atop \mathcal{D} = \mathcal{E} \dot\cup \mathcal{F} \text{ is prefix-free}} & = & |\Delta_P^n(\mathcal{E}) \cap \Delta_S^n(\mathcal{D})| + |\mathcal{F} \cap \Delta_S^n(\mathcal{D})| \\
{\scriptstyle \mathcal{F} \subseteq \mathcal{D} \text{ and } \mathcal{D} \text{ is suffix-free}} & = & |\Delta_P^n(\mathcal{E}) \cap \Delta_S^n(\mathcal{D})| + |\mathcal{F}| \\
{\scriptstyle \text{equation (4.13)}} & = & \frac{k}{q} |\Delta_P^n(\mathcal{E})| + |\mathcal{F}| \\
 & \geq & \frac{k}{q} \cdot (|\Delta_P^n(\mathcal{E})| + |\mathcal{F}|) = \frac{k}{q} \cdot |\Delta_P^n(\mathcal{D})| \\
{\scriptstyle \text{with (4.10) for } m=0} & = & \frac{k}{q} \cdot q^{n-1} \cdot k = k^2 \cdot q^{n-2} .
\end{array}$$

With the above equation and (4.10) for $m = 0$ follows:

$$\begin{array}{rcccl}
|\Delta_B^n(\mathcal{D})| & = & |\Delta_P^n(\mathcal{D})| + |\Delta_S^n(\mathcal{D})| - |\Delta_P^n(\mathcal{D}) \cap \Delta_S^n(\mathcal{D})| & \leq & 2\,|\Delta_P^n(\mathcal{D})| - q^{n-2} \cdot k^2 \\
 & = & 2q^{n-1}k - q^{n-2}k^2 & = & q^{n-2}k(2q - k)
\end{array}$$

Thus:

$$|\Delta_B^n(\mathcal{D})| = q^{n-2}k(2q-k)\,. \tag{4.14}$$

Since the Kraftsum of $(\alpha_n)_{n\in\mathbb{N}}$ is smaller than or equal to γ_k and $\mathcal{D}$ is a $\pi_q(\alpha_1,\ldots,\alpha_{n-1},\beta;k)$-system, we obtain by (4.10):

$$\gamma_k \cdot q^n \geq \sum_{l=1}^{n} \alpha_l q^{n-l} = |\Delta_P^n(\mathcal{D})| + (\alpha_n - \beta) = k\cdot q^{n-1} + (\alpha_n-\beta)\,.$$

It follows:

$$\alpha_n - \beta \leq q^n\Big(\gamma_k - \frac{k}{q}\Big)\,. \tag{4.15}$$

By (4.14) and (4.15)we obtain:

$$|\Delta_B^n(\mathcal{D})| + (\alpha_n-\beta) \leq q^n\Big(\frac{k(2q-k)}{q^2} + \gamma_k - \frac{k}{q}\Big)\,. \tag{4.16}$$

¿From (4.6) follows, that the term inside paranthesis of the right-hand side of equation (4.16) is smaller than or equal to one.

$$\begin{array}{rl}
(4.6) \Rightarrow & \gamma_k \leq \left(\frac{q-k}{q}\right)^2 + \frac{k}{q} \\
\Leftrightarrow & q^2\gamma_k \leq (q-k)^2 + kq = q^2 - 2kq + k^2 + kq \\
\Leftrightarrow & k(2q-k) + q^2\gamma_k - kq \leq q^2 \\
\Leftrightarrow & \frac{k(2q-k)}{q^2} + \gamma_k - \frac{k}{q} \leq 1\,.
\end{array}$$

Therefore we conclude:

$$|\Delta_B^n(\mathcal{D})| + (\alpha_n-\beta) \leq q^n = |\mathcal{A}^n|\,.$$

This shows, that we can choose $(\alpha_n-\beta)$ codewords $c_1,\ldots c_{\alpha_n-\beta} \in \mathcal{A}^n - \Delta_B^n(\mathcal{D})$ of length n which are not in the bifix-shadow of $\mathcal{D}$. Then $\mathcal{C}_0 := \mathcal{D}\cup\{c_1,\ldots,c_{\alpha_n-\beta}\}$ is a fix-free code which extend $\mathcal{D}$, whereas $\left|\mathcal{C}_0\cap\mathcal{A}^l\right| = \alpha_l$ for all $1\leq l\leq n$.

m → m+1 :
Let $\mathcal{C}_m$ be a fix-free extension of $\mathcal{D}$ which fits to $\alpha_1, \ldots, \alpha_{n+m}$. More precisely :

> $\mathcal{C}_m$ with $\mathcal{D} \subseteq \mathcal{C}_m \subseteq \bigcup_{l=1}^{n+m} \mathcal{A}^l$ is fix-free and $\left|\mathcal{C}_m \cap \mathcal{A}^l\right| = \alpha_l$ for all $1 \leq l \leq n+m$.

We will show that there exists α_{n+m+1} codewords of length $(n+m+1)$ which are not in the bifix-shadow of $\mathcal{C}_m$. If we add this codewords to $\mathcal{C}_m$ we obtain a fix-free extension $\mathcal{C}_{m+1}$ of $\mathcal{D} \subseteq \mathcal{C}_m$ which fits to $(\alpha_1, \ldots, \alpha_{n+m+1})$.

We define $\mathcal{X}$ and M as:

$$\mathcal{X} := \mathcal{C}_m - \mathcal{D} \quad ; \quad M := n+m+1.$$

Because $\mathcal{C}_m$ is fix-free, $\mathcal{C}_m = \mathcal{X} \cup \mathcal{D}$ and $\mathcal{X} \cap \mathcal{D} = \emptyset$ we obtain:

$$\begin{aligned}
\left|\Delta_B^M(\mathcal{C}_m)\right| &= \left|\Delta_B^M(\mathcal{X}) \cup \Delta_B^M(\mathcal{D})\right| \\
&= \left|\Delta_B^M(\mathcal{D})\right| + \left|\Delta_B^M(\mathcal{X})\right| - \left|\Delta_B^M(\mathcal{X}) \cap \Delta_B^M(\mathcal{D})\right| \\
&= 2\left|\Delta_P^M(\mathcal{D})\right| + 2\left|\Delta_P^M(\mathcal{X})\right| - \left|\Delta_P^M(\mathcal{D}) \cap \Delta_S^M(\mathcal{D})\right| - \left|\Delta_P^M(\mathcal{X}) \cap \Delta_S^M(\mathcal{X})\right| \\
&\quad - \left|\Delta_B^M(\mathcal{X}) \cap \Delta_B^M(\mathcal{D})\right| \\
&\leq 2\left|\Delta_P^M(\mathcal{D})\right| + 2\left|\Delta_P^M(\mathcal{X})\right| - \left|\Delta_P^M(\mathcal{D}) \cap \Delta_S^M(\mathcal{D})\right| - \left|\Delta_B^M(\mathcal{X}) \cap \Delta_B^M(\mathcal{D})\right| .
\end{aligned} \tag{4.17}$$

For the terms in the sum on the right-hand side of inequality (4.17) we get:

$$\left|\Delta_P^M(\mathcal{D})\right| = q^{M-1}k\,. \tag{4.18}$$

This follows from (4.10).

$$\left|\Delta_P^M(\mathcal{D}) \cap \Delta_S^M(\mathcal{D})\right| = q^{M-2}k^2 \tag{4.19}$$

Whereas the above equation follows from:

$$\begin{aligned}
\left|\Delta_P^M(\mathcal{D}) \cap \Delta_S^M(\mathcal{D})\right| &= \left|\bigcup_{i,j=1}^{k} \left(\Delta_P^M(\mathcal{D}_i) \cap \Delta_S^M(\mathcal{D}_j)\right)\right| \\
\text{\small $\mathcal{D}$ is fix-free and the disjoint union of the D_i} \quad &= \sum_{i,j=1}^{k} \left|\Delta_P^M(\mathcal{D}_i) \cap \Delta_S^M(\mathcal{D}_j)\right| \\
&= \sum_{i,j=1}^{k} \left|\Delta_P^{M-1}(\mathcal{D}_i)\mathcal{A} \cap \Delta_S^M(\mathcal{D}_j)\right| \\
\text{\small (4.9) and Lemma 51 (i)} \quad &= \sum_{i,j=1}^{k} \left|\Delta_P^{M-1}(\mathcal{D}_i)\right| \\
\text{\small with (4.8)} \quad &= q^{M-2}k^2\,.
\end{aligned}$$

Let us determine the value of $\left|\Delta_B^M(\mathcal{X}) \cap \Delta_B^M(\mathcal{D})\right|$. We have :

$$\Delta_B^M(\mathcal{X}) \cap \Delta_B^M(\mathcal{D}) = (\Delta_P^M(\mathcal{X}) \cap \Delta_S^M(\mathcal{D})) \,\dot{\cup}\, (\Delta_S^M(\mathcal{X}) \cap \Delta_P^M(\mathcal{D}))\,.$$

This holds, because $\mathcal{C}_m$ is fix-free and the union of $\mathcal{D}$ and $\mathcal{X}$ is disjoint. Therefore we get $\Delta_S^M(\mathcal{X}) \cap \Delta_S^M(\mathcal{D}) = \Delta_P^M(\mathcal{X}) \cap \Delta_P^M(\mathcal{D}) = \emptyset$. Furthermore we have $\Delta_P^M(\mathcal{D}) \cap \Delta_P^M(\mathcal{X}) = \emptyset$. It follows:

$$\begin{aligned}\left|\Delta_B^M(\mathcal{X}) \cap \Delta_B^M(\mathcal{D})\right| &= \left|\Delta_P^M(\mathcal{X}) \cap \Delta_S^M(\mathcal{D})\right| + \left|\Delta_S^M(\mathcal{X}) \cap \Delta_P^M(\mathcal{D})\right| \\ &= \left|\Delta_P^{M-1}(\mathcal{X})\mathcal{A} \cap \Delta_S^M(\mathcal{D})\right| + \left|\mathcal{A}\Delta_S^{M-1}(\mathcal{X}) \cap \Delta_P^M(\mathcal{D})\right|\,.\end{aligned}$$

$\mathcal{D} = \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_k$ is fix-free and the union of the $\mathcal{D}_i$ is disjoint. Therefore from the above equation follows:

$$\left|\Delta_B^M(\mathcal{X}) \cap \Delta_B^M(\mathcal{D})\right| = \sum_{i=1}^{k} \left(\left|\Delta_P^{M-1}(\mathcal{X})\mathcal{A} \cap \Delta_S^M(\mathcal{D}_i)\right| + \left|\mathcal{A}\Delta_S^{M-1}(\mathcal{X}) \cap \Delta_P^M(\mathcal{D}_i)\right| \right). \tag{4.20}$$

By (4.8), (4.9) and Lemma 51 (i) for all $1 \leq i \leq k$ follows:

$$\begin{aligned}\left|\Delta_P^{M-1}(\mathcal{X})\mathcal{A} \cap \Delta_S^M(\mathcal{D}_i)\right| &= \left|\Delta_P^{M-1}(\mathcal{X})\right| = \frac{\left|\Delta_P^M(\mathcal{X})\right|}{q} \quad \text{and} \\ \left|\mathcal{A}\Delta_S^{M-1}(\mathcal{X}) \cap \Delta_P^M(\mathcal{D}_i)\right| &= \left|\Delta_S^{M-1}(\mathcal{X})\right| = \left|\Delta_P^{M-1}(\mathcal{X})\right| = \frac{\left|\Delta_P^M(\mathcal{X})\right|}{q}\,.\end{aligned}$$

¿From the above equations and (4.20) we obtain:

$$\left|\Delta_B^M(\mathcal{X}) \cap \Delta_B^M(\mathcal{D})\right| = \frac{2k}{q}\left|\Delta_P^M(\mathcal{X})\right|\,. \tag{4.21}$$

For the cardinality of $\Delta_B^M(\mathcal{C}_m)$ we obtain from (4.17), (4.18) ,(4.19) and (4.21):

$$\left|\Delta_B^M(\mathcal{C}_m)\right| \leq k \cdot \frac{2q-k}{q^2}q^M - \frac{2k}{q}\left|\Delta_P^M(\mathcal{X})\right| + 2\left|\Delta_P^M(\mathcal{X})\right|\,. \tag{4.22}$$

We have $\Delta_P^M(\mathcal{X}) = \Delta_P^M(\mathcal{C}_m - \mathcal{D}) = \Delta_P^M(\mathcal{C}_m) - \Delta_P^M(\mathcal{D})$, because $\mathcal{C}_m$ is fix-free and $\mathcal{D} \subseteq \mathcal{C}_m$. If we take into account that $\mathcal{C}_m$ fits to $\alpha_1, \ldots, \alpha_{m+n}$ (whereas $m + n = M - 1$) and the Kraftsum of $(\alpha_l)_{l\in\mathbb{N}}$ is smaller than or equal to $\frac{k}{q}$, we obtain that the following equalities and inequalities are true.

$$
\begin{aligned}
\left|\Delta_P^M(\mathcal{X})\right| + \alpha_M &= \left|\Delta_P^M(\mathcal{C}_m)\right| + \alpha_M - \left|\Delta_P^M(\mathcal{D})\right| \overset{(4.10)}{=} \left|\Delta_P^M(\mathcal{C}_m)\right| + \alpha_M - q^{M-1}k \\
&= \sum_{l=1}^{M-1} \alpha_l q^{M-l} + \alpha_M - q^{M-1}k = \Big(\sum_{l=1}^{M} \alpha_l q^{-l} - \frac{k}{q}\Big) \cdot q^M \\
&\leq \Big(\gamma_k - \frac{k}{q}\Big) \cdot q^M .
\end{aligned}
$$

$$
\Rightarrow \left|\Delta_P^M(\mathcal{X})\right| \leq \left|\Delta_P^M(\mathcal{X})\right| + \alpha_M \leq \Big(\gamma_k - \frac{k}{q}\Big) \cdot q^M
$$

By the above equation and (4.22) follows:

$$
\begin{aligned}
\left|\Delta_B^M(\mathcal{C}_m)\right| &\leq k \cdot \frac{2q-k}{q^2} q^M - \frac{2k}{q}\Big(\gamma_k - \frac{k}{q}\Big) \cdot q^M + 2\Big(\gamma_k - \frac{k}{q}\Big) \cdot q^M \\
&= \Big(\frac{k(2q-k)}{q^2} + \Big(2 - \frac{2k}{q}\Big)\Big(\gamma_k - \frac{k}{q}\Big)\Big) \cdot q^M .
\end{aligned}
$$

¿From (4.6) follows that the term inside the big paranthesis on the right hand side of the above equation is smaller than or equal to one.

$$
\begin{aligned}
(4.6) \Rightarrow \quad & \gamma_k \leq \tfrac{1}{2} + \tfrac{k}{2q} \\
\Leftrightarrow \quad & 2q\gamma_k - 2k \leq q - k \\
{\scriptstyle (0 \leq k < q)} \Leftrightarrow \quad & (2q\gamma_k - 2k)(q-k) \leq (q-k)^2 \\
\Leftrightarrow \quad & 2qk - k^2 + (2q - 2k)(q\gamma_k - k) \leq q^2 \\
\Leftrightarrow \quad & \tfrac{k(2q-k)}{q^2} + \Big(2 - \tfrac{2k}{q}\Big)\Big(\gamma_k - \tfrac{k}{q}\Big) \leq 1
\end{aligned}
$$

Therefore we conclude:

$$
\left|\Delta_B^M(\mathcal{C}_m)\right| + \alpha_{n+m+1} = \left|\Delta_B^M(\mathcal{C}_m)\right| + \alpha_M \leq q^M = \left|\mathcal{A}^{n+m+1}\right| .
$$

This shows that we can add α_{n+m+1} codewords of length $(n+m+1)$ to $\mathcal{C}_m$, which are not in the bifix-shadow of $\mathcal{C}_m$. In this way we obtain a fix-free code $\mathcal{C}_{m+1}$ with $\mathcal{C}_{m+1} \supseteq \mathcal{C}_m \supseteq \mathcal{D}$ and $\left|\mathcal{C}_{m+1} \cap \mathcal{A}^l\right| = \alpha_l$ for all $1 \leq l \leq n+m+1$.

Let $\mathcal{C} := \bigcup_{l=0}^{\infty} \mathcal{C}_l$. Because of $\mathcal{D} \subseteq \mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2 \ldots$, the set $\mathcal{C}$ is fix-free and $\left|\mathcal{C}_{m+1} \cap \mathcal{A}^l\right| = \alpha_l$ for all $l \geq 1$. Therefore $\mathcal{C}$ is a fix-free extension of $\mathcal{D}$ which fits to $(\alpha_l)_{l\in\mathbb{N}}$. **q.e.d**

Corollary 3 *Let $(\alpha_l)_{l\in\mathbb{N}}$ a sequence of nonnegative integers with*

$$\lceil\frac{q}{2}\rceil\frac{1}{q} < \sum_{l=1}^{\infty}\alpha_l q^{-l} \le \frac{3}{4}$$

and $n \in \mathbb{N}$, $1 \le \beta \le \alpha_n$ such that:

$$\beta q^{-n} + \sum_{l=1}^{n-1}\alpha_l q^{-l} = \left\lceil\frac{q}{2}\right\rceil\frac{1}{q}$$

Then for every $\pi_q(\alpha_1,\ldots,\alpha_{n-1},\beta,\lceil\frac{q}{2}\rceil)$-system there exists a fix-free extension which fits $(\alpha_l)_{l\in\mathbb{N}}$.

Proof : We have to show that $\gamma_{\lceil\frac{q}{2}\rceil} \ge \frac{3}{4}$ for all $q \ge 2$.
For even q we obtain:

$$\gamma_{\lceil\frac{q}{2}\rceil} = \frac{1}{2} + \frac{\frac{q}{2}}{2q} = \frac{3}{4}.$$

For odd q let $q = 2t+1$:

$$\begin{array}{rl} & \gamma_{\lceil\frac{q}{2}\rceil} = \left(\frac{q-\lceil\frac{q}{2}\rceil}{q}\right)^2 + \frac{\lceil\frac{q}{2}\rceil}{q} = \frac{\lfloor\frac{q}{2}\rfloor^2 + q\lceil\frac{q}{2}\rceil}{q^2} \ge \frac{3}{4} \\ \Leftrightarrow & 4\left\lfloor\frac{q}{2}\right\rfloor^2 + 4q\left\lceil\frac{q}{2}\right\rceil \ge 3q^2 \\ \Leftrightarrow & 4t^2 + 4(2t+1)(t+1) \ge 3(2t+1)^2 \\ \Leftrightarrow & 12t^2 + 12t + 4 \ge 12t^2 + 12t + 3 \\ \Leftrightarrow & 4 \ge 3 \end{array}$$

Therefore $\gamma_{\lceil\frac{q}{2}\rceil} \ge \frac{3}{4}$ for all $q \ge 2$. The corollary follows from Theorem 16. **q.e.d**

For the binary case $|\mathcal{A}| = 2$ we obtain from the corollary Lemma 47.

The table below shows the values from γ_k for $q \in \{2,3,4,5,6\}$

$q\backslash k$	1	2	3	4	5
2	$\frac{3}{4}$				
3	$\frac{2}{3}$	$\frac{7}{9}$			
4	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{13}{16}$		
5	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{19}{25}$	$\frac{21}{25}$	
6	$\frac{7}{12}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{7}{9}$	$\frac{31}{36}$

Next we give some easy example fors π-systems:

Example 13 Let $|\mathcal{A}| = q \geq 2$ and $k, d \in \mathbb{N}$ such that $1 \leq d < q$ and $k \leq \min\{d, q-d\}$. Furthermore let $\mathcal{X}, \mathcal{Y}$ be a partition of $\mathcal{A}$ with

$$|\mathcal{X}| = d \qquad ; \qquad |\mathcal{Y}| = q - d\,.$$

Since $k \leq \min\{d, q-d\}$ we can choose permutations of $\mathcal{X}$ $\varphi_1, \ldots, \varphi_k : \mathcal{X} \leftrightarrow \mathcal{X}$ and permutations of $\mathcal{Y}$ $\phi_1, \ldots, \phi_k : \mathcal{Y} \leftrightarrow \mathcal{Y}$ with the property:

$$\varphi_i(x) \neq \varphi_j(x) \;\text{ and }\; \phi_i(y) \neq \phi_j(y) \quad \text{for all }\; i \neq j\,,\; x \in \mathcal{X}\,,\; y \in \mathcal{Y} \tag{4.23}$$

For example, let $\mathcal{X} = \{x_0, \ldots x_{d-1}\}$ and $\mathcal{Y} = \{y_0, \ldots, y_{q-d-1}\}$ then it is possible to choose the φ_i and the ϕ_i as

$$\varphi_i(x_l) := x_{l+i-1 \bmod d} \;\text{ and }\; \phi_i(y_m) := y_{m+i-1 \bmod q-d}$$

for all $1 \leq i \leq k$, $0 \leq l \leq d-1$, $0 \leq m \leq q-d-1$.

For $1 \leq i \leq k$ and $n \geq 2$ we define:

$$\mathcal{D}_i := \bigcup_{y \in \mathcal{Y}} y\mathcal{Y}^{n-2}\phi_i(y) \;\cup\; \bigcup_{m=0}^{n-2} \bigcup_{x \in \mathcal{X}} x\mathcal{Y}^m \varphi_i(x)\,,$$

$$\mathcal{D} := \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_k\,.$$

By (4.23) the sets $\mathcal{D}_1, \ldots, \mathcal{D}_k$ are pairwise disjoint and $\mathcal{D}$ is fix-free, because $\mathcal{X}, \mathcal{Y}$ is a partition of $\mathcal{D}$. While the φ_i's are permutations of $\mathcal{X}$ and the ϕ_i's are permutations of $\mathcal{Y}$, we obtain for $1 \leq i \leq k$:

$$\mathcal{A}^{-1}\mathcal{D}_i \;=\; \mathcal{Y}^{n-1} \cup \bigcup_{m=0}^{n-2} \mathcal{Y}^m \mathcal{X} \quad ; \quad \mathcal{D}_i\mathcal{A}^{-1} \;=\; \mathcal{Y}^{n-1} \cup \bigcup_{m=0}^{n-2} \mathcal{X}\mathcal{Y}^m\,.$$

It follows:

$$|\mathcal{A}^{-1}\mathcal{D}_i| = |\mathcal{D}_i\mathcal{A}^{-1}| = |\mathcal{D}_i| = (q-d)^{n-1} + \sum_{m=0}^{n-2} d \cdot (q-d)^m\,.$$

Obviously $\mathcal{A}^{-1}\mathcal{D}_i$ is prefix-free and $\mathcal{D}_i\mathcal{A}^{-1}$ is suffix-free. The equation below shows, that they are maximal, too.

$$\begin{aligned} S(\mathcal{A}^{-1}\mathcal{D}_i) = S(\mathcal{D}_i\mathcal{A}^{-1}) &= (q-d)^{n-1} \cdot q^{-n+1} + \sum_{l=1}^{n-1} d \cdot (q-d)^{l-1} \cdot q^{-l} \\ &= (\tfrac{q-d}{q})^{n-1} + \tfrac{d}{q} \cdot \sum_{l=0}^{n-2} (\tfrac{q-d}{q})^{l} \\ &= (\tfrac{q-d}{q})^{n-1} + \tfrac{d}{q} \cdot \frac{1-\left(\frac{q-d}{q}\right)^{n-1}}{1-\frac{q-d}{q}} \\ &= (\tfrac{q-d}{q})^{n-1} + \tfrac{d}{q} \cdot \frac{1-\left(\frac{q-d}{q}\right)^{n-1}}{\frac{d}{q}} \\ &= (\tfrac{q-d}{q})^{n-1} + 1 - (\tfrac{q-d}{q})^{n-1} = 1 \end{aligned}$$

Therefore $\mathcal{A}^{-1}\mathcal{D}$ is maximal prefix-free and $\mathcal{D}\mathcal{A}^{-1}$ is maximal suffix-free. This shows that(3) in Definition 5 holds for $\mathcal{D}_1, \ldots, \mathcal{D}_k$. Therefore $\mathcal{D}$ is a $\pi_q(n;k)$-system.

For the numbers of codewords of length l we obtain:

$$\begin{aligned} |\mathcal{D} \cap \mathcal{A}| &= |\mathcal{D} \cap \mathcal{A}^l| = 0 & &\text{for } l > n \geq 2\,, \\ |\mathcal{D} \cap \mathcal{A}^l| &= k \cdot d \cdot (q-d)^{l-2} & &\text{for } 2 \leq l < n\,, \\ |\mathcal{D} \cap \mathcal{A}^n| &= k \cdot d \cdot (q-d)^{n-2} + k \cdot (q-d)^{n-1} = k \cdot q \cdot (q-d)^{n-2} & &. \end{aligned}$$

Therefore by Theorem 16 we obtain the following proposition:

Proposition 52 *Let* $|\mathcal{A}| = q \geq 2$, $n \geq 2$, $1 \leq d < q$, $k \leq \min\{d, q-d\}$ *and* $(\alpha_l)_{l\in\mathbb{N}}$ *be a sequence of nonnegative integers with* $\sum_{l=1}^{\infty} \alpha_l \cdot q^{-l} \leq \gamma_k$ *whereat* γ_k *is chosen as in Theorem 16. If* $\alpha_1 = 0$, $\alpha_l = k \cdot d \cdot (q-d)^{l-2}$ *for* $2 \leq l < n$ *and* $\alpha_n \geq k \cdot q \cdot (q-d)^{n-2}$ *then there exists a fix-free code which fits* $(\alpha_l)_{l\in\mathbb{N}}$.

For even q we can choose $k = d = q - d = \frac{q}{2}$. Because of $\gamma_{\frac{q}{2}} = \frac{3}{4}$, we obtain in this case:

Proposition 53 *Let* $|\mathcal{A}| = q$ *with* q *even,* $n \geq 2$ *and* $(\alpha_l)_{l\in\mathbb{N}}$ *be a sequence of nonnegative integers with* $\sum_{l=1}^{\infty} \alpha_l \cdot q^{-l} \leq \frac{3}{4}$ *If* $\alpha_1 = 0$, $\alpha_l = (\frac{q}{2})^l$ *for* $2 \leq l < n$ *and* $\alpha_n \geq q \cdot (\frac{q}{2})^{n-1}$ *then there exists a fix-free code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$.

For the binary case we conclude:

Proposition 54 *Let* $|\mathcal{A}| = 2$ *and* $(\alpha_l)_{l\in\mathbb{N}}$ *be a sequence of nonnegative integers with* $\sum_{l=1}^{\infty} \alpha_l \cdot q^{-l} \leq \frac{3}{4}$. *If there exists an* $n \geq 2$ *such that* $\alpha_0 = 0$, $\alpha_l = 1$ *for* $2 \leq l < n$ *and* $\alpha_n \geq 2$ *then there exists a fix-free code* $\mathcal{C} \subseteq \mathcal{A}^+$ *which fits to* $(\alpha_l)_{l\in\mathbb{N}}$.

Example 14 Let $\mathcal{A} := \{0, \ldots, q-1\}$ for some $q \geq 2$. We will show that for any $n \in \mathbb{N}$ and $1 \leq k < q$ there exist one level $\pi_q(n;k)$-systems. If $n = 1$ then we can choose $\mathcal{D} := \{0, \ldots, k-1\}$ and $\mathcal{D}_i := \{i\}$ for $0 \leq i \leq k-1$. Then $\mathcal{D}$ is a $\pi_q(1;k)$-system with π-partition $\mathcal{D}_0, \ldots, \mathcal{D}_{k-1}$. Thus let us assume that $n \geq 2$. We choose permutations $\varphi_0, \ldots, \varphi_{k-1} : \mathcal{A} \leftrightarrow \mathcal{A}$ with the property:

$$\varphi_i(a) \neq \varphi_j(a) \text{ for all } i \neq j\,, \quad a \in \mathcal{A} \tag{4.24}$$

For example, if we choose $\varphi_i(a) := a + i \bmod q$ for all $a \in \mathcal{A}$ and $0 \leq i \leq k-1$, then $\varphi_0, \ldots, \varphi_{k-1}$ are permutations of $\mathcal{A}$ for which (4.24) holds.

We define the sets $\mathcal{D}_0, \ldots, \mathcal{D}_{k-1} \subseteq \mathcal{A}^n$ as:

$$\mathcal{D}_i := \bigcup_{a=0}^{q-1} a\mathcal{A}^{n-2}\varphi_i(a) \subseteq \mathcal{A}^n \text{ for } 0 \leq i \leq k-1$$

Because of (4.24) the sets $\mathcal{D}_0, \ldots, \mathcal{D}_{k-1}$ are pairwise disjoint. They are a partition of $\mathcal{D}$, where

$$\mathcal{D} := \bigcup_{i=0}^{k-1} \mathcal{D}_i \subseteq \mathcal{A}^n.$$

Furthermore $\mathcal{D}$ is fix-free, because it is a subset of $\mathcal{A}^n$.

Because $\varphi_0, \ldots, \varphi_{k-1}$ are permutations of $\mathcal{A}$ we obtain that for any $0 \leq i \leq k-1$ the sets $A^{n-2}\varphi_i(0), \ldots, \mathcal{A}^{n-2}\varphi_i(q-1)$ are a partition of $\mathcal{A}^{n-1}$ For all $0 \leq i \leq k-1$ follows:

$$\begin{aligned}
|\mathcal{D}_i| &= \sum_{a=0}^{q-1} |a\mathcal{A}^{n-2}\varphi_i(a)| = q \cdot q^{n-2} = q^{n-1}, \\
|\mathcal{A}^{-1}\Delta_P^n(\mathcal{D}_i)| &= |\mathcal{A}^{-1}\mathcal{D}_i| = |\bigcup_{a=0}^{q-1} \mathcal{A}^{n-2}\varphi_i(a)| = |\mathcal{A}^{n-1}| = q^{n-1}, \\
|\Delta_S^n(\mathcal{D}_i)\mathcal{A}^{-1}| &= |\mathcal{D}_i\mathcal{A}^{-1}| = |\bigcup_{a=0}^{q-1} a\mathcal{A}^{n-2}| = |\mathcal{A}^{n-1}| = q^{n-1}.
\end{aligned}$$

Therefore (1) in the Definition of π-systems holds. This means $\mathcal{D}$ is a one level $\pi_q(n;k)$-system with π-partition $\mathcal{D}_0, \ldots, \mathcal{D}_{k-1}$. This shows that for every $n \in \mathbb{N}$ and $1 \leq k < q$, there exists a $\pi_q(n;k)$-system $\mathcal{D}$ with $\mathcal{D} \subseteq \mathcal{A}^n$.

By Theorem 16 and Corollary 3 we conclude that the following proposition holds:

Proposition 55 *Let $|\mathcal{A}| = q \geq 2$, $1 \leq k < q$, γ_k as in Theorem 16 and $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers.*

(i): If $\sum\limits_{l=1}^{\infty} \alpha_l \cdot q^{-l} \leq \gamma_k$, $\alpha_1 = \ldots = \alpha_{n-1} = 0$ and $\alpha_n \geq \frac{k}{q}$ for some $n \in \mathbb{N}$, then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits $(\alpha_l)_{l\in\mathbb{N}}$.

(ii): If $\sum\limits_{l=1}^{\infty} \alpha_l \cdot q^{-l} \leq \frac{3}{4}$, $\alpha_1 = \ldots = \alpha_{n-1} = 0$ and $\alpha_n \geq \left\lceil \frac{q}{2} \right\rceil \frac{1}{q}$ for some $n \in \mathbb{N}$, then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits $(\alpha_l)_{l\in\mathbb{N}}$.

For the binary case we obtain:

Proposition 56 *Let $|\mathcal{A}| = 2$ and $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers with If $\sum\limits_{l=1}^{\infty} \alpha_l \cdot \left(\frac{1}{2}\right)^2 \leq \frac{3}{4}$, $\alpha_1 = \ldots = \alpha_{n-1} = 0$ and $\alpha_n \geq \frac{1}{2}$ for some $n \in \mathbb{N}$, then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits $(\alpha_l)_{l\in\mathbb{N}}$.*

Example 15 Let $|\mathcal{A}| = q \geq 2$ and $k, d \in \mathbb{N}$ such that $1 \leq d < q$ and $k \leq \min\{d, q-d\}$. Furthermore let $\mathcal{X}, \mathcal{Y}$ a partition of $\mathcal{A}$ with

$$|\mathcal{X}| = d \qquad ; \qquad |\mathcal{Y}| = q - d$$

As in Example 13 we can choose permutations $\varphi_1, \ldots, \varphi_k : \mathcal{X} \leftrightarrow \mathcal{X}$ and permutations $\phi_1, \ldots, \phi_k : \mathcal{Y} \leftrightarrow \mathcal{Y}$ with (4.23).

For $n \geq 3$ and $1 \leq i \leq k$ we define the sets $\mathcal{D}_i, \mathcal{E}_i, \mathcal{F}_i, \mathcal{G}_i$ and $\mathcal{D}$ as:

$$\begin{aligned}
\mathcal{E}_i &:= \bigcup_{l=1}^{n-2} \bigcup_{x\in\mathcal{X}} x\mathcal{Y}^l \varphi_i(x) \\
\mathcal{F}_i &:= \bigcup_{l=1}^{n-2} \bigcup_{y\in\mathcal{Y}} y\mathcal{X}^l \phi_i(y) \\
\mathcal{G}_i &:= \bigcup_{x\in\mathcal{X}} x\mathcal{X}^{n-2}\varphi_i(x) \;\cup\; \bigcup_{y\in\mathcal{Y}} y\mathcal{Y}^{n-2}\phi_i(y) \\
\mathcal{D}_i &:= \mathcal{E}_i \cup \mathcal{F}_i \cup \mathcal{G}_i \\
\mathcal{D} &= \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_k
\end{aligned}$$

Obviously $\mathcal{D}$ is fix-free. The permutations $\varphi_1, \ldots \varphi_k$ and $\phi_1, \ldots, \phi_k$ fulfill (4.23). Therefore $\mathcal{D}_1, \ldots, \mathcal{D}_k$ are pairwise disjoint. For $1 \leq i \leq k$ the sets $\mathcal{E}_i, \mathcal{F}_i, \mathcal{G}_i$ are pairwise disjoint and φ_i and ϕ_i are permutations of $\mathcal{X}$ and $\mathcal{Y}$. It follows:

$$|\mathcal{D}_i| = |\mathcal{E}_i| + |\mathcal{G}_i| + |\mathcal{F}_i| = d \cdot \sum_{l=1}^{n-2} (q-d)^l + (q-d) \cdot \sum_{l=1}^{n-2} d^l + d^{n-1} + (q-d)^{n-1}. \quad (4.25)$$

Because φ_i and ϕ_i are permutations of $\mathcal{X}$ and $\mathcal{Y}$ we obtain:

$$\begin{array}{rclcrcl}
\mathcal{A}^{-1}\mathcal{D}_i &=& \mathcal{A}^{-1}\mathcal{E}_i \cup \mathcal{A}^{-1}\mathcal{F}_i \cup \mathcal{A}^{-1}\mathcal{G}_i &,& \mathcal{D}_i\mathcal{A}^{-1} &=& \mathcal{E}_i\mathcal{A}^{-1} \cup \mathcal{F}_i\mathcal{A}^{-1} \cup \mathcal{G}_i\mathcal{A}^{-1}, \\
\mathcal{A}^{-1}\mathcal{E}_i &=& \bigcup_{l=1}^{n-2} \mathcal{Y}^l\mathcal{X} &,& \mathcal{E}_i\mathcal{A}^{-1} &=& \bigcup_{l=1}^{n-2} \mathcal{X}\mathcal{Y}^l, \\
\mathcal{A}^{-1}\mathcal{F}_i &=& \bigcup_{l=1}^{n-2} \mathcal{X}^l\mathcal{Y} &,& \mathcal{F}_i\mathcal{A}^{-1} &=& \bigcup_{l=1}^{n-2} \mathcal{Y}\mathcal{X}^l.
\end{array}$$

$$\mathcal{A}^{-1}\mathcal{G}_i = \mathcal{G}_i\mathcal{A}^{-1} = \mathcal{X}^{n-1} \cup \mathcal{Y}^{n-1}$$

Obviously $\mathcal{A}^{-1}\mathcal{D}_i$ is prefix-free and $\mathcal{D}_i\mathcal{A}^{-1}$ is suffix-free for $n \geq 3$. Furthermore $\mathcal{A}^{-1}\mathcal{E}_i, \mathcal{A}^{-1}\mathcal{G}_i, \mathcal{A}^{-1}\mathcal{F}_i$ are pairwise disjoint and also $\mathcal{E}_i\mathcal{A}^{-1}, \mathcal{F}_i\mathcal{A}^{-1}, \mathcal{G}_i\mathcal{A}^{-1}$ are pairwise disjoint. Therefore

$$|\mathcal{A}^{-1}\mathcal{D}_i| = |\mathcal{A}^{-1}\mathcal{E}_i| + |\mathcal{A}^{-1}\mathcal{F}_i| + |\mathcal{A}^{-1}\mathcal{G}_i| = d \cdot \sum_{l=1}^{n-2} (q-d)^l + (q-d) \cdot \sum_{l=1}^{n-2} d^l + d^{n-1} + (q-d)^{n-1}.$$

The same way follows $|\mathcal{D}_i\mathcal{A}^{-1}| = d \cdot \sum_{l=1}^{n-2} (q-d)^l + (q-d) \cdot \sum_{l=1}^{n-2} d^l + d^{n-1} + (q-d)^{n-1}$.
By (4.25) follows:

$$|\mathcal{A}^{-1}\mathcal{D}_i| = |\mathcal{D}_i\mathcal{A}^{-1}| = |\mathcal{D}_i|.$$

Let us show that $\mathcal{A}^{-1}\mathcal{D}_i$ and $\mathcal{D}_{\mathcal{A}}^{-1}$ are maximal as well.

$$S(\mathcal{A}^{-1}\mathcal{G}_i) = S(\mathcal{G}_i\mathcal{A}^{-1}) = \left(\frac{d}{q}\right)^{n-1} + \left(\frac{q-d}{q}\right)^{n-1}$$

$$\begin{aligned}
S(\mathcal{A}^{-1}\mathcal{E}_i) &= S(\mathcal{E}_i\mathcal{A}^{-1}) = \sum_{l=2}^{n-1} d \cdot (q-d)^{l-1} \cdot q^{-l} = d \cdot \sum_{l=0}^{n-3} (q-d)^{l+1} \cdot q^{-l-2} \\
&= \tfrac{d \cdot (q-d)}{q^2} \cdot \sum_{l=0}^{n-3} \left(\tfrac{q-d}{q}\right)^l = \tfrac{d \cdot (q-d)}{q^2} \cdot \frac{1-(\frac{q-d}{q})^{n-2}}{1-\frac{q-d}{q}} = \tfrac{q-d}{q} - (\tfrac{q-d}{q})^{n-1}
\end{aligned}$$

Same way:

$$S(\mathcal{A}^{-1}\mathcal{F}_i) = S(\mathcal{F}_i\mathcal{A}^{-1}) = \frac{d}{q} - \left(\frac{d}{q}\right)^{n-1}$$

$\mathcal{A}^{-1}\mathcal{E}_i$ $\mathcal{A}^{-1}\mathcal{F}_i$ $\mathcal{A}^{-1}\mathcal{G}_i$ and $\mathcal{E}_i\mathcal{A}^{-1}$, $\mathcal{F}_i\mathcal{A}^{-1}$, $\mathcal{G}_i\mathcal{A}^{-1}$ are pairwise disjoint, therefore:

$$\begin{aligned}
S(\mathcal{A}^{-1}\mathcal{D}_i) &= S(\mathcal{A}^{-1}\mathcal{E}_i) + S(\mathcal{A}^{-1}\mathcal{F}_i) + S(\mathcal{A}^{-1}\mathcal{G}_i) \\
&= \tfrac{q-d}{q} - \left(\tfrac{q-d}{q}\right)^{n-1} + \tfrac{d}{q} - \left(\tfrac{d}{q}\right)^{n-1} + \left(\tfrac{d}{q}\right)^{n-1} + \left(\tfrac{q-d}{q}\right)^{n-1} = 1
\end{aligned}$$

In the same way follows:

$$S(\mathcal{D}\mathcal{A}^{-1}) = S(\mathcal{E}_i\mathcal{A}^{-1}) + S(\mathcal{F}_i\mathcal{A}^{-1}) + S(\mathcal{G}_i\mathcal{A}^{-1}) = 1\,.$$

This shows, that $\mathcal{A}^{-1}\mathcal{D}_i$ is maximal prefix-free and $\mathcal{D}\mathcal{A}^{-1}$ maximal suffix-free. Therefore $\mathcal{D}$ is a $\pi_q(n;k)$-system for all $n \geq 3$.
The numbers of codewords of length l is given by:

$$\begin{array}{lcll}
|\mathcal{D} \cap \mathcal{A}^l| & = & 0 & \text{for } \; l > n \geq 3 \; \text{ or } \; l \in \{1, 2\} \\
|\mathcal{D} \cap \mathcal{A}^l| & = & k \cdot (d \cdot (q-d)^{l-2} + (q-d) \cdot d^{l-2}) & \text{for } \; 3 \leq l < n \\
|\mathcal{D} \cap \mathcal{A}^n| & = & k \cdot q \cdot ((q-d)^{n-2} + d^{n-2}) &
\end{array}$$

Similar like in example 1 we obtain with Theorem 16 the following propositions:

Proposition 57 *Let* $|\mathcal{A}| = q \geq 2$*,* $n \geq 3$*,* $1 \leq d < q$*,* $k \leq \min\{d, q-d\}$ *and* $(\alpha_l)_{l\in\mathbb{N}}$ *be a sequence of nonnegative integers with* $\sum_{l=1}^{\infty} \alpha_l \cdot q^{-l} \leq \gamma_k$ *where* γ_k *is chosen as in Theorem 16. If* $\alpha_1 = \alpha_2 = 0$*,* $\alpha_l = k \cdot (d \cdot (q-d)^{l-2} + (q-d) \cdot d^{l-2})$ *for* $3 \leq l < n$ *and* $\alpha_n \geq k \cdot q \cdot ((q-d)^{n-2} + d^{n-2})$ *then there exists a fix-free code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

For q even and $k = d = q - d = \frac{q}{2}$ we obtain:

Proposition 58 *Let* $|\mathcal{A}| = q$ *with* q *even,* $n \geq 3$ *and* $(\alpha_l)_{l\in\mathbb{N}}$ *be a sequence of nonnegative integers with* $\sum_{l=1}^{\infty} \alpha_l \cdot q^{-l} \leq \frac{3}{4}$ *If* $\alpha_1 = \alpha_2 = 0$*,* $\alpha_l = 2 \cdot (\frac{q}{2})^l$ *for* $3 \leq l < n$ *and* $\alpha_n \geq 2q(\frac{q}{2})^{n-1}$ *then there exists a fix-free code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

Finally we obtain for the binary case:

Proposition 59 *Let* $|\mathcal{A}| = 2$ *and* $(\alpha_l)_{l\in\mathbb{N}}$ *be a sequence of nonnegative integers with* $\sum_{l=1}^{\infty} \alpha_l \cdot q^{-l} \leq \frac{3}{4}$*. If there exists an* $n \geq 2$ *such that* $\alpha_0 = 0$*,* $\alpha_l = 2$ *for* $3 \leq l < n$ *and* $\alpha_n \geq 4$ *then there exists a fix-free code* $\mathcal{C} \subseteq \mathcal{A}^+$ *which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

4.2 Generation of π -systems by regular subgraphs of $\mathcal{B}_q(n)$

Lemma 60 *Let $|\mathcal{A}| = q \geq 2$ and $\mathcal{X} \subseteq \mathcal{A}^n$ for some $n \geq 1$. Then:*
$|\mathcal{A}^{-1}\mathcal{X}| = |\mathcal{X}\mathcal{A}^{-1}| = |\mathcal{A}\mathcal{X} \cap \mathcal{X}\mathcal{A}| = |\mathcal{X}|$ *if and only if $\mathcal{X}$ is the edge-set of a 1-regular subgraph in $\mathcal{B}_q(n-1)$*

Proof: Let $\mathcal{X} \subseteq \mathcal{A}^n$ be the edge set of a 1-regular subgraph $\Lambda := (\mathcal{V}, \mathcal{X}) \subseteq (\mathcal{A}^{n-1}, \mathcal{A}^n)$ in $\mathcal{B}_q(n-1)$, where $\mathcal{V}$ should denote the vertex set of Λ. The set $\mathcal{X}\mathcal{A}^{-1}$ is the set of vertices which are initial vertices of some edge in $\mathcal{X}$, and the set $\mathcal{A}^{-1}\mathcal{X}$ is the set of vertices, which are terminal vertices of some edge in $\mathcal{X}$. Since $\mathcal{X}$ is the edge set of a regular subgraph in $\mathcal{B}_q(n-1)$, it follows that:

$$\mathcal{A}^{-1}\mathcal{X} = \mathcal{V} = \mathcal{X}\mathcal{A}^{-1}\,.$$

For a 1-regular graph the number of vertices is equal to the number of edges, therefore we obtain:

$$|\mathcal{X}| = |\mathcal{V}| = |\mathcal{A}^{-1}\mathcal{X}| = |\mathcal{X}\mathcal{A}^{-1}|\,.$$

Λ is 1-regular and therefore for every $v \in \mathcal{V}$ there exist a unique edge in $\mathcal{X}$ which is incident to v and a unique edge in $\mathcal{X}$ which is incident from v. It follows, that for every $v \in \mathcal{V}$ there exist unique $a, b \in \mathcal{A}$ with $av, vb \in \mathcal{X}$ and then also $avb \in \mathcal{A}\mathcal{X} \cap \mathcal{X}\mathcal{A}$ holds. This shows, that there exists a one-to-one map $G : \mathcal{V} \to \mathcal{A}\mathcal{X} \cap \mathcal{X}\mathcal{A}$.

Let $w \in \mathcal{A}\mathcal{X} \cap \mathcal{X}\mathcal{A}$, then there exist $a, b \in \mathcal{A}$ and $v \in \mathcal{A}^{n-1}$ with $av, vb \in \mathcal{X}$ and $w = avb$. While av is an edge in Λ, we obtain $v \in \mathcal{V}$ and it follows, that $G(v) = avb = w$. This shows, that G is a bijection and therefore we obtain:

$$|\mathcal{X}| = |\mathcal{V}| = |\mathcal{A}\mathcal{X} \cap \mathcal{X}\mathcal{A}|\,.$$

This shows the first part of the lemma.

Thus let us show the other direction of the lemma. Let $\mathcal{X} \subseteq \mathcal{A}^n$ be a set with $|\mathcal{X}| = |\mathcal{A}^{-1}\mathcal{X}| = |\mathcal{X}\mathcal{A}^{-1}| = |\mathcal{A}\mathcal{X} \cap \mathcal{X}\mathcal{A}|$. Moreover let $\Lambda = (\mathcal{V}, \mathcal{X})$ be the subgraph (without isolated vertices) of $\mathcal{B}_q(n-1)$ with edge-set $\mathcal{X}$. We have to show that Λ is one-regular.

First we have:

$$|(\mathcal{A}\mathcal{X} \cap \mathcal{X}\mathcal{A})\mathcal{A}^{-1}| = |\mathcal{A}\mathcal{X} \cap \mathcal{X}\mathcal{A}|\,. \tag{4.26}$$

To show (4.26), let us assume that $|(\mathcal{A}\mathcal{X}\cap\mathcal{X}\mathcal{A})\mathcal{A}^{-1}| < |\mathcal{A}\mathcal{X}\cap\mathcal{X}\mathcal{A}|$. Then there exist $a,b,c\in\mathcal{A}$ and $w\in\mathcal{A}^{n-1}$ with $a\neq b$ and $cwa, cwb\in\mathcal{A}\mathcal{X}\cap\mathcal{X}\mathcal{A}$. It follows, that $wa, wb\in\mathcal{X}$. This is a contradiction, because $|\mathcal{X}| = |\mathcal{X}\mathcal{A}^{-1}|$.

In the same way we obtain:

$$|\mathcal{A}^{-1}(\mathcal{A}\mathcal{X}\cap\mathcal{X}\mathcal{A})| = |\mathcal{A}\mathcal{X}\cap\mathcal{X}\mathcal{A}|\,. \tag{4.27}$$

¿From $\mathcal{A}^{-1}(\mathcal{A}\mathcal{X}\cap\mathcal{X}\mathcal{A})$, $(\mathcal{A}\mathcal{X}\cap\mathcal{X}\mathcal{A})\mathcal{A}^{-1}\subseteq\mathcal{X}$, $|\mathcal{A}\mathcal{X}\cap\mathcal{X}\mathcal{A}| = |\mathcal{X}|$, (4.26) and (4.27) follows:

$$\begin{aligned} \mathcal{X} &= \mathcal{A}^{-1}(\mathcal{A}\mathcal{X}\cap\mathcal{X}\mathcal{A}) = \mathcal{A}^{-1}\mathcal{X}\mathcal{A}\cap\mathcal{X} && (4.28)\\ &= (\mathcal{A}\mathcal{X}\cap\mathcal{X}\mathcal{A})\mathcal{A}^{-1} = \mathcal{A}\mathcal{X}\mathcal{A}^{-1}\cap\mathcal{X}\,. && (4.29)\end{aligned}$$

Let $bv\in\mathcal{X}$ with $b\in\mathcal{A}$ and $v\in\mathcal{A}^{n-1}$. ¿From (4.28) follows that, there exists a letter $a\in\mathcal{A}$ with $va\in\mathcal{X}$ and from $|\mathcal{X}\mathcal{A}^{-1}| = |\mathcal{X}|$ follows, that the letter a is unique. Furthermore $v\in\mathcal{V}$ because bv is an edge in Λ. Thus we have:

> Let v be a vertex of Λ, such that there is at least one edge of Λ with terminal vertex v. Then there exists an unique edge of Λ with initial vertex v. (4.30)

In the same way we obtain from (4.44) and $|\mathcal{A}^{-1}\mathcal{X}| = |\mathcal{X}|$:

> Let v be a vertex of Λ, such that there is at least one edge of Λ with initial vertex v. Then there exists an unique edge of Λ with terminal vertex v. (4.31)

¿From (4.30) and (4.31) follows, that Λ is 1-regular. **q.e.d**

Theorem 17 *Let $|\mathcal{A}| = q \geq 2$, $n \geq 1$ and $1 \leq k < q$.*

(i) Let $\mathcal{D}$ be a two level $\pi_q(n+1;k)$-system with $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ or a one level $\pi_q(n;k)$-system with $\mathcal{D} \subseteq \mathcal{A}^n$. Then there exists $1 \leq L \leq q^{n-1}$ such that for any π-partition $\mathcal{D}_1, \ldots, \mathcal{D}_k$ of $\mathcal{D}$:

$$\begin{array}{lllllllll} L & = & |\mathcal{D}_1 \cap \mathcal{A}^n| & = & |\mathcal{D}_2 \cap \mathcal{A}^n| & = & \ldots & = & |\mathcal{D}_k \cap \mathcal{A}^n| \\ q^n - Lq & = & |\mathcal{D}_1 \cap \mathcal{A}^{n+1}| & = & |\mathcal{D}_2 \cap \mathcal{A}^{n+1}| & = & \ldots & = & |\mathcal{D}_k \cap \mathcal{A}^{n+1}| \end{array}$$

i.e., $|\mathcal{D} \cap \mathcal{A}^n| = kL$, $|\mathcal{D} \cap \mathcal{A}^{n+1}| = kq(q^{n-1} - L)$ and $\mathcal{D}$ is a one level π-system iff $L = q^{n-1}$

(ii) Let $1 \leq L < q^{n-1}$, then there exists a two level $\pi_q(n;k)$-system $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n-1}$ with $kL = |\mathcal{D} \cap \mathcal{A}^n|$ if and only if there exists a k-regular subgraph in $\mathcal{B}_q(n-1)$ with L vertices.

(iii) $\mathcal{D} \subseteq \mathcal{A}^n$ is a (one level) $\pi_q(n;k)$-system with π-partition $\mathcal{D}_1, \ldots, \mathcal{D}_k$ if and only if $\mathcal{D}$ is the edge set of a k-factor Λ in $\mathcal{B}_q(n-1)$ and $\mathcal{D}_1, \ldots, \mathcal{D}_k$ are the edge sets of an edge disjoint decomposition of Λ into 1-factors.

¿From (i) follows, that there exists two level $\pi_q(n+1;k)$-systems $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ only of the form $|\mathcal{D} \cap \mathcal{A}^n| = kL$ for some $1 \leq L < q^{n-1}$. As the proof of the theorem will show any such π-system and any π-partition for two level π-systems can be constructed as described below.

Construction 1

1. Let $\Lambda := (\mathcal{V}, \mathcal{X}) \subseteq (\mathcal{A}^n, \mathcal{A}^{n+1})$ be a k-regular proper subgraph of $\mathcal{B}_q(n-1)$ with $L = |\mathcal{V}|$.
2. Choose a decomposition of Λ into k edge disjoint 1-factors $\Lambda_1, \ldots, \Lambda_k$ of Λ. Let $\mathcal{X}_i$ denote the edge set of the Λ_i for all $1 \leq i \leq k$.
3. Choose permutations $\varphi_1, \ldots, \varphi_k : \mathcal{A} \longleftrightarrow \mathcal{A}$ with the property:

$$\varphi_i(a) \neq \varphi_j(a) \quad \forall\, a \in \mathcal{A}\,,\; i \neq j$$

   and define

$$\begin{array}{lll} \mathcal{V}^c & := & \mathcal{A}^{n-1} - \mathcal{V}\,, \\ \mathcal{Y}_i & := & \bigcup_{a \in \mathcal{A}} a\mathcal{V}^c\varphi_i(a) \quad \forall\, 1 \leq i \leq k\,, \\ \mathcal{Y} & := & \mathcal{Y}_1 \cup \ldots \cup \mathcal{Y}_k\,. \end{array}$$

4. Let $\mathcal{D} := \mathcal{X} \cup \mathcal{Y} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ and $\mathcal{D}_i := \mathcal{X}_i \cup \mathcal{Y}_i$ for all $1 \leq i \leq k$

$\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n-1}$ is a two level $\pi_q(n+1;k)$-system with $|\mathcal{D} \cap \mathcal{A}^n| = kL$ and π-partition $\mathcal{D}_1, \ldots, \mathcal{D}_k$. Furthermore any such $\mathcal{D}$ and π-partition $\mathcal{D}_1, \ldots, \mathcal{D}_k$ of $\mathcal{D}$ can constructed in such a way.

If $\mathcal{A} = \{0, \ldots, q-1\}$, we can choose the permutations $\varphi_1, \ldots, \varphi_k$ in step 3 as:

$$\varphi_i(a) := a + i - 1 \bmod q \quad \text{for all } \ a \in \mathcal{A}\,, \quad 1 \leq i \leq k\,. \tag{4.32}$$

If one needs only the π-system $\mathcal{D}$ without a certain π-partition the following construction is possible. Let $\mathcal{X} \subseteq \mathcal{A}^n$ be the edge set of a k-regular subgraph $\Lambda \subseteq \mathcal{B}_q(n-1)$ with L vertices and $\mathcal{A}_a := \{\varphi_1(a), \ldots \varphi_k(a)\}$, where $\varphi_1, \ldots, \varphi_k$ are permutations with the property in step 3. For example, let $\mathcal{A} := \{0, \ldots, q-1\}$ and the φ_i as in (4.32), then $\mathcal{A}_a = \{a \bmod q, (a+1) \bmod q, \ldots, (a+k-1) \bmod q\}$. We define $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n-1}$ as:

$$\mathcal{D} \cap \mathcal{A}^n := \mathcal{X} \ \text{ and } \ \mathcal{D} \cap \mathcal{A}^{n+1} := \bigcup_{a \in \mathcal{A}} a(\mathcal{A}^{n-1} - \mathcal{A}^{-1}\mathcal{X})\mathcal{A}_a\,.$$

Then $\mathcal{D}$ is a two level $\pi_q(n+1;k)$-system, because $\mathcal{A}^{-1}\mathcal{X} = \mathcal{X}\mathcal{A}^{-1}$ is the vertex set of Λ and therefore $\mathcal{D} \cap \mathcal{A}^{n+1}$ is the same as the set $\mathcal{Y}$ in step 3. (i.e., $\mathcal{A}^{-1}\mathcal{X}$ are the vertices of Λ which has at least one antecessor vertex in Λ and $\mathcal{X}\mathcal{A}^{-1}$ are vertices which have at least one successor vertex in Λ.)

For a given two level $\pi_q(n+1;k)$-system $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ neither the decomposition of Λ into 1-factors in step 2, nor the permutations $\varphi_1, \ldots, \varphi_k$ in step 3 are unique. The above construction shows, that $\mathcal{D}$ has in general more than one π-partition. In the same way by Theorem 17 (iii) follows, that an one level π-systems has more than one π-partition, because regular subgraphs in $\mathcal{B}_q(n-1)$ has in general more than one decomposition into edge disjoint 1-factors.

Proof of Theorem 17 : Let $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ a be two level $\pi_q(n+1;k)$-system or a one level $\pi_q(n;k)$-system and $\mathcal{D}_1, \ldots, \mathcal{D}_k$ a π-partition of $\mathcal{D}$. We define:

$$\begin{array}{lll} \mathcal{X} := \mathcal{D} \cap \mathcal{A}^n & , \quad \mathcal{Y} := \mathcal{D} \cap \mathcal{A}^{n+1}\,, & \\ \mathcal{X}_i := \mathcal{D}_i \cap \mathcal{A}^n & , \quad \mathcal{Y}_i := \mathcal{D}_i \cap \mathcal{A}^{n+1} & \text{for } 1 \leq i \leq k\,, \\ \text{and } L_i := |\mathcal{X}_i| = |\mathcal{D}_i \cap \mathcal{A}^n| & & \text{for } 1 \leq i \leq k\,. \end{array}$$

Claim 1 *$\mathcal{X}$ is the edge set of a k-regular subgraph in $\mathcal{B}_q(n-1)$ and $\mathcal{X}_1,\ldots,\mathcal{X}_k$ are the edge sets of edge disjoint 1-factors of this subgraph.*

¿From the properties of π-partition follows for all $1 \le i \le k$:

$$q^n = |\Delta_P^{n+1}(D_i)| = |\mathcal{X}_i\mathcal{A}| + |\mathcal{Y}_i| = qL_i + |\mathcal{Y}_i|$$

and therefore we have

$$|\mathcal{Y}_i| = q^n - qL_i \quad \text{for all} \quad 1 \le i \le k\,. \tag{4.33}$$

Since the $\mathcal{D}_i$'s are fix-free and
$|\Delta_P^{n+1}(\mathcal{D}_i)| = |\Delta_S^{n+1}(\mathcal{D}_i)| = |\mathcal{A}^{-1}\Delta_P^{n+1}(\mathcal{D}_i)| = |\Delta_S^{n+1}(\mathcal{D}_i)\mathcal{A}^{-1}|$, by Lemma 50 follows:

$$L_i = |\mathcal{X}_i| = |\mathcal{X}_i\mathcal{A}^{-1}| = |\mathcal{A}^{-1}\mathcal{X}_i| \text{ and } q^n - qL_i = |\mathcal{Y}_i| = |\mathcal{Y}_i\mathcal{A}^{-1}| = |\mathcal{A}^{-1}\mathcal{Y}_i|\,. \tag{4.34}$$

While $\mathcal{A}^{-1}\mathcal{D}_i$ is prefix-free and $\mathcal{D}_i\mathcal{A}^{-1}$ is suffix-free, it follows that
$\mathcal{A}^{-1}\mathcal{X}_i\mathcal{A} \cap \mathcal{A}^{-1}\mathcal{Y} = \mathcal{A}\mathcal{X}_i\mathcal{A}^{-1} \cap \mathcal{Y}\mathcal{A}^{-1} = \emptyset$. Thus we obtain:

$$\mathcal{A}(\mathcal{X}_i\mathcal{A}^{-1})\mathcal{A} \cap (\mathcal{Y}_i\mathcal{A}^{-1})\mathcal{A} = \mathcal{A}(\mathcal{A}^{-1}\mathcal{X}_i)\mathcal{A} \cap \mathcal{A}(\mathcal{A}^{-1}\mathcal{Y}_i) = \emptyset \tag{4.35}$$

Furthermore we have $\mathcal{A}\mathcal{X}_i \cap \mathcal{A}(\mathcal{A}^{-1}\mathcal{Y}_i) = \emptyset$, because $\mathcal{D}_i$ is suffix-free. Therefore we obtain:

$$\begin{aligned}
q^{n+1} &\ge |\mathcal{A}(\mathcal{A}^{-1}\mathcal{X}_i)\mathcal{A} \cup \mathcal{A}\mathcal{X}_i \cup \mathcal{A}(\mathcal{A}^{-1}\mathcal{Y}_i)| \\
&= |\mathcal{A}(\mathcal{A}^{-1}\mathcal{X}_i)\mathcal{A}| + |\mathcal{A}\mathcal{X}_i| + |\mathcal{A}(\mathcal{A}^{-1}\mathcal{Y}_i)| - |\mathcal{A}(\mathcal{A}^{-1}\mathcal{X}_i)\mathcal{A} \cap \mathcal{A}\mathcal{X}_i| \\
&= q^2L_i + qL_i + q(q^n - qL_i) - |\mathcal{A}(\mathcal{A}^{-1}\mathcal{X}_i)\mathcal{A} \cap \mathcal{A}\mathcal{X}_i| \\
&= qL_i + q^{n+1} - |\mathcal{A}(\mathcal{A}^{-1}\mathcal{X}_i)\mathcal{A} \cap \mathcal{A}\mathcal{X}_i|\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
& qL_i = |\mathcal{A}\mathcal{X}_i| \ge |\mathcal{A}(\mathcal{A}^{-1}\mathcal{X}_i)\mathcal{A} \cap \mathcal{A}\mathcal{X}_i| \ge qL_i \\
\Rightarrow\ & |\mathcal{A}(\mathcal{A}^{-1}\mathcal{X}_i)\mathcal{A} \cap \mathcal{A}\mathcal{X}_i| = qL_i \\
\Rightarrow\ & |\mathcal{A}^{-1}\mathcal{X}_i\mathcal{A} \cap \mathcal{X}_i| = L_i = |\mathcal{X}_i| \\
\Rightarrow\ & \mathcal{A}^{-1}\mathcal{X}_i\mathcal{A} \cap \mathcal{X}_i = \mathcal{X}_i\,.
\end{aligned}$$

¿From the last equation follows, that for every $x_1 \ldots x_n \in \mathcal{X}_i$ there exists a letter $a \in \mathcal{X}_i$ such that $x_2 \ldots x_n a \in \mathcal{X}_i$. For this letter we have
$x_1 \ldots x_n a \in \mathcal{A}\mathcal{X}_l \cap \mathcal{X}_i\mathcal{A} \subseteq \mathcal{A}\mathcal{X}_i$. By $|\mathcal{X}_i\mathcal{A}^{-1}| = |\mathcal{X}_i|$ we obtain, that the letter a is

unique. This shows, that there exists a one-to-one map from $\mathcal{X}_i$ into $\mathcal{A}\mathcal{X}_i \cap \mathcal{X}_i\mathcal{A}$. Furthermore this map is a bijection, because for every $w_1 \dots w_n w_{n+1} \in \mathcal{A}\mathcal{X}_i \cap \mathcal{X}_i\mathcal{A} \subseteq \mathcal{X}_i\mathcal{A}$ we have $w_1 \dots w_n \in \mathcal{X}_i$. Thus we conclude:

$$|\mathcal{X}_i| = |\mathcal{A}\mathcal{X}_i \cap \mathcal{X}_i\mathcal{A}| \quad \forall\, 1 \leq i \leq k\,. \tag{4.36}$$

By Lemma 60 follows that each $\mathcal{X}_i$ is the edge set of a 1-regular subgraph $\Lambda_i := (\mathcal{V}_i, \mathcal{X}_i) \subseteq (\mathcal{A}^n, \mathcal{A}^{n+1})$ of $\mathcal{B}_q(n-1)$, where we denote by $\mathcal{V}_i$ the vertex set of Λ_i. For $1 \leq i, j \leq k$ with $i \neq j$. We claim:

Claim 2 *For every $x \in \mathcal{X}_j$ with $x = x_1 \dots x_n$, $x_1, \dots x_n \in \mathcal{A}$ there exists an unique letter $x_1' \in \mathcal{A}$ with $x_1 \neq x_1'$ and $x_1' x_2 \dots x_n \in \mathcal{X}_i$.*

Let $x = x_1 \dots x_n \in \mathcal{X}_j$ be as in the claim. From $|\Delta_S^{n+1}(\mathcal{D}_i)\mathcal{A}^{-1}| = |\Delta_S^{n+1}(\mathcal{D}_i)| = q^n$ and Lemma 51 (i) follows:

$$1 = |\{x\}| = |x\mathcal{A} \cap \Delta_S^{n+1}(\mathcal{D}_i)| = |(\mathcal{A}\mathcal{X}_i \cup \mathcal{Y}_i) \cap x\mathcal{A}| = |\mathcal{A}\mathcal{X}_i \cap x\mathcal{A}| + |\mathcal{Y}_i \cap x\mathcal{A}|\,. \tag{4.37}$$

Since $\mathcal{D}_i \cup \mathcal{D}_j$ is prefix-free, we obtain $|\mathcal{Y}_i \cap x\mathcal{A}| = 0$ and it follows that $|\mathcal{A}\mathcal{X}_i \cap x\mathcal{A}| = 1$. Thus we find $a, b \in \mathcal{A}$ and $z_1 \dots z_n \in \mathcal{X}_i$, $z_1, \dots, z_n \in \mathcal{A}$ with $a z_1 \dots z_n = x_1 \dots x_n b$. While $\mathcal{X}_i$ is the edge set of a 1-regular subgraph in $\mathcal{B}_q(n-1)$, the edge $z_1 \dots z_n$ has a unique antecessor edge in $\mathcal{X}_i$. It means that there exists an unique letter $x_1' \in \mathcal{A}$ with $x_1' z_1 \dots z_{n-1} \in \mathcal{X}_i$. It follows that $x_1' x_2 \dots x_n \in \mathcal{X}_i$. By $\mathcal{X}_i \cap \mathcal{X}_j \subseteq \mathcal{D}_i \cap \mathcal{D}_j = \emptyset$, we obtain that $x_1' \neq x_1$. Furthermore from $|\mathcal{A}^{-1}\mathcal{X}_i| = |\mathcal{X}_i|$ it follows that there is no other $c \neq x_1'$, $c \in \mathcal{A}$ with $c x_2 \dots x_n \in \mathcal{X}_i$. This shows Claim 2.

¿From Claim 2 follows :

$$\mathcal{A}^{-1}\mathcal{X}_i = \mathcal{A}^{-1}\mathcal{X}_j \quad \forall\, 1 \leq i, j \leq k\,. \tag{4.38}$$

By (4.34) we have:

$$L_i = |\mathcal{X}_i| = |\mathcal{A}^{-1}\mathcal{X}_i| = |\mathcal{A}^{-1}\mathcal{X}_j| = |\mathcal{X}_j| = L_j \quad \forall\, 1 \leq i, j \leq k\,. \tag{4.39}$$

Thus let

$$L := L_1 = \dots = L_k\,. \tag{4.40}$$

Furthermore we have $\mathcal{V}_i = \mathcal{A}^{-1}\mathcal{X}_i$ for all $1 \leq i \leq k$, because all Λ_i's are 1-regular graphs and therefore every vertex $v \in \mathcal{V}_i$ is the terminal vertex of a unique edge in $\mathcal{X}_i$. Thus from (4.38) follows, that all Λ_i's have the same vertex set. We define:

$$\mathcal{V} := \mathcal{V}_1 = \dots = \mathcal{V}_k \;\text{ and }\; \Lambda := \bigcup_{i=1}^{k} \Lambda_i = (\mathcal{V}, \mathcal{X}_1 \cup \dots \cup \mathcal{X}_k) = (\mathcal{V}, \mathcal{X})\,.$$

Especially we obtain:

$$\mathcal{V} = \mathcal{A}^{-1}\mathcal{X}_i = \mathcal{X}_i\mathcal{A}^{-1} = \mathcal{X}\mathcal{A}^{-1} = \mathcal{A}^{-1}\mathcal{X} \quad \forall\, 1 \le i \le k\,. \tag{4.41}$$

While the edge sets $\mathcal{X}_1, \ldots, \mathcal{X}_k$ are pairwise disjoint, it follows that Λ is the union of the k edge disjoint 1-regular graphs $\Lambda_1, \ldots, \Lambda_k$ having all the same vertex set. Therefore Λ is a k-regular subgraph of $\mathcal{B}_q(n-1)$ with $|\mathcal{V}| = L$ vertices and $\Lambda_1, \ldots, \Lambda_k$ is an edge disjoint decomposition of Λ into 1-factors. This shows Claim 1.

Furthermore from (4.39), (4.40) and (4.33) follows part (i) of the theorem.

If $\mathcal{D}$ is a one level π-system then $\mathcal{D} = \mathcal{X}\,,\quad \mathcal{Y} = \emptyset$ and $\mathcal{V} = \mathcal{A}^{n-1}$. In this case Λ is a k-factor and $\Lambda_1, \ldots, \Lambda_k$ are 1-factors of $\mathcal{B}_q(n-1)$ and from Claim 1 follows the "only if" part of Theorem 17 (iii). If $\mathcal{D}$ is a two level π-system, then from Claim 1 follows the "only if " part of Theorem 17 (ii).

In the case that $\mathcal{D}$ is a two level π-system, we show, that $\mathcal{D}$ and $\mathcal{D}_1, \ldots \mathcal{D}_k$ can be constructed as described in Construction 1.

Claim 3 *There exist (unique) permutations $\varphi_1, \ldots, \varphi_k : \mathcal{A} \longleftrightarrow \mathcal{A}$ with the property*

$$\varphi_i(a) \neq \varphi_j(a) \quad \forall\, a \in \mathcal{A}\,,\ i \neq j$$

such that $\mathcal{Y}_i = \bigcup\limits_{a\in\mathcal{A}} a(\mathcal{A}^{n-1} - \mathcal{V})\varphi_i(a)$ for all $1 \le i \le k$.

Let us assume that there exist $a, b \in \mathcal{A}$ and $v \in \mathcal{V}$ with $avb \in \mathcal{Y}_i$ for some $1 \le i \le k$. Then $vb \in \mathcal{A}^{-1}\mathcal{Y}_i$ and by (4.41) we have $v \in \mathcal{A}^{-1}\mathcal{X}_i$. This is a contradiction, because $\mathcal{D}$ is a π-system, i.e., $\mathcal{A}^{-1}\mathcal{D}_i = \mathcal{A}^{-1}\mathcal{X}_i \cup \mathcal{A}^{-1}\mathcal{Y}_i$ is prefix-free. Therefore we obtain:

$$\mathcal{Y}_i \subseteq \mathcal{A}(\mathcal{A}^{n-1} - \mathcal{V})\mathcal{A} \ \text{ for all } \ 1 \le i \le k\,. \tag{4.42}$$

By (4.33) follows:

$$|\mathcal{Y}_i| = q^n - qL = q(q^{n-1} - |\mathcal{V}|) = |\mathcal{A}| \cdot |\mathcal{A}^{n-1} - \mathcal{V}|\,. \tag{4.43}$$

Let $a, b, c \in \mathcal{A}$ and $w \in \mathcal{A}^{n-1} - \mathcal{V}$ such that $awb\,,\, awc \in \mathcal{Y}_i$. Because of $\mathcal{Y}_i = \mathcal{D}_i \cap \mathcal{A}^{n+1}$ and $\mathcal{D}$ is a π-system follows from Lemma 50, that $|\mathcal{Y}_i| = |\mathcal{A}^{-1}\mathcal{Y}_i|$ holds. This shows $c = b$. It follows, that for any $a \in \mathcal{A}$ and $w \in \mathcal{A}^{n-1} - \mathcal{V}$ with $aw \in \mathcal{Y}_i\mathcal{A}^{-1}$ there exists a unique $b \in \mathcal{A}$ with $awb \in \mathcal{Y}_i$. With (4.42) and (4.43) follows, that there exists a map $\varphi_i : \mathcal{A} \longrightarrow \mathcal{A}$ such that $\mathcal{Y}_i = \bigcup\limits_{a\in\mathcal{A}} a(\mathcal{A}^{n-1} - \mathcal{V})\varphi_i(a)$.

Obviously the map φ_i is unique. To show that φ_i is a bijection we take into account that from Lemma 50 also follows, that $|\mathcal{Y}_i| = |\mathcal{Y}_i\mathcal{A}^{-1}|$. This means that for any $w \in \mathcal{A}^{n-1} - \mathcal{V}$ and $b \in \mathcal{A}$ with $wb \in \mathcal{Y}_i\mathcal{A}^{-1}$ there exists a unique $a \in \mathcal{A}$ such that $awb \in \mathcal{Y}_i$. This shows that φ_i is a one-to-one map, i.e. a permutation of the alphabet $\mathcal{A}$. Furthermore the sets $\mathcal{Y}_1, \ldots, \mathcal{Y}_k$ are pairwise disjoint and therefore $\varphi_i(a) \neq \varphi_j(a)$ for all $a \in \mathcal{A}$ and $i \neq j$. This shows that Claim 3 holds. By Claim 1 follows, that any two level π-system $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n-1}$ and any π-partition of $\mathcal{D}$ is of the form described in Construction 1.

We finish the proof, by showing that the set $\mathcal{D}$ in Construction 1 is a π-system with π-partition $\mathcal{D}_1, \ldots, \mathcal{D}_k$ and that any edge set of a k-factor of $\mathcal{B}_q(n-1)$ is a one level $\pi_q(n;k)$-system, whereas a π-partition is given by the edge sets of an edge disjoint decomposition of the k-factor into 1-factors. This shows the other direction of Theorem 17 (ii) and (iii).

Let $\Lambda := (\mathcal{V}, \mathcal{E}) \subseteq (\mathcal{A}^{n-1}, \mathcal{A}^n)$ be a k-regular subgraph of $\mathcal{B}_q(n-1)$ and let $L := |\mathcal{V}|$. By Proposition 26 we obtain, that there are k edge disjoint 1-factors $\Lambda_1, \ldots, \Lambda_k$ of Λ, i.e. Λ is the edge disjoint union of the Λ_i's. Let $\mathcal{X}_1, \ldots, \mathcal{X}_k$ be the edge sets of $\Lambda_1, \ldots, \Lambda_k$. Then

$$|\mathcal{X}_i| = |\mathcal{V}| = L \text{ and } \mathcal{X}_i \cap \mathcal{X}_j = \emptyset \quad \forall\, 1 \leq i, j \leq k\,.$$

With Lemma 60 we obtain:

$$|\mathcal{A}\mathcal{X}_i \cap \mathcal{X}_i\mathcal{A}| = |\mathcal{A}^{-1}\mathcal{X}_i| = |\mathcal{X}_i\mathcal{A}^{-1}| = |\mathcal{X}_l| = |\mathcal{V}| = L \quad \forall\, 1 \leq i \leq k\,. \tag{4.44}$$

Let $\mathcal{V}^c := \mathcal{A}^{n-1} - \mathcal{V}$ and $\varphi_1, \ldots, \varphi_k : \mathcal{A} \leftrightarrow \mathcal{A}$ be permutations of $\mathcal{A}$ with the property:

$$\varphi_i(a) \neq \varphi_j(a) \quad \forall\, a \in \mathcal{A} \;\text{ and }\; i \neq j\,. \tag{4.45}$$

We define for all $1 \leq i \leq k$:

$$\begin{array}{rcl} \mathcal{Y}_i & := & \bigcup_{a \in \mathcal{A}} a\mathcal{V}^c\varphi_i(a)\,, \\ \mathcal{D}_i & := & \mathcal{X}_i \cup \mathcal{Y}_i\,, \\ \mathcal{Y} & := & \mathcal{Y}_1 \cup \ldots \cup \mathcal{Y}_k \subseteq \mathcal{A}\mathcal{V}^c\mathcal{A}\,, \\ \mathcal{D} & := & \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_k = \mathcal{X} \cup \mathcal{Y}\,. \end{array}$$

¿From (4.44) follows:

$$|\mathcal{D}| = |\mathcal{X}| + |\mathcal{Y}| = L + |\mathcal{A}(\mathcal{A}^{n-1} - \mathcal{V})| = L + q^n - qL\,. \tag{4.46}$$

For any subgraph of $\mathcal{B}_q(n-1)$ with vertex set $\mathcal{V} \subseteq \mathcal{A}^{n-1}$ and edge set $\mathcal{E} \subseteq \mathcal{A}^n$, the sets $\mathcal{V}\mathcal{A}$ and $\mathcal{A}\mathcal{V}$ are subsets of $\mathcal{E}$. Therefore we obtain for all $1 \leq i \leq k$:

$$\mathcal{A}\mathcal{V} \subseteq \mathcal{X} \text{ and } \mathcal{V}\mathcal{A} \subseteq \mathcal{X} . \tag{4.47}$$

It follows, that $\mathcal{D} = \mathcal{X} \cup \mathcal{Y}$ is fix-free. Furthermore by property (4.45) of the φ_i's follows, that $\mathcal{Y}_1, \ldots, \mathcal{Y}_k$ are pairwise disjoint and therefore $\mathcal{D}_1, \ldots, D_k$ is a partition of $\mathcal{D}$. While the φ_i's are permutations of $\mathcal{A}$ we have for all $1 \leq i \leq k$

$$\mathcal{A}^{-1}\mathcal{Y}_i = (\mathcal{A}^{n-1} - \mathcal{V})\mathcal{A} \text{ and } \mathcal{Y}_i\mathcal{A}^{-1} = \mathcal{A}(\mathcal{A}^{n-1} - \mathcal{V}) . \tag{4.48}$$

All Λ_i's are 1-regular subgraphs with vertex set $\mathcal{V}$, i.e. for every $v \in \mathcal{V}$ there is an edge in $\mathcal{X}_i$ incident to v and an edge incident from v. It follows, that:

$$\mathcal{V} = \mathcal{A}^{-1}\mathcal{X}_i = \mathcal{X}_i\mathcal{A}^{-1} \quad \text{for all } 1 \leq i \leq k . \tag{4.49}$$

By (4.45), (4.48) and (4.49) we obtain for all $1 \leq i \leq k$:

$$|\mathcal{A}^{-1}\mathcal{D}_i| = |\mathcal{A}^{-1}\mathcal{X}_i| + |\mathcal{A}^{-1}\mathcal{Y}_i| = L + q^n - qL = |\mathcal{D}_i| .$$

In the same way $|\mathcal{D}_i\mathcal{A}^{-1}| = |\mathcal{D}_i|$ can be shown for all $1 \leq i \leq k$.
Furthermore from (4.48) and (4.49) follows, that $\mathcal{A}^{-1}\mathcal{D}_i$ is prefix-free and $\mathcal{D}_i\mathcal{A}^{-1}$ is suffix-free. For the Kraftsum of $\mathcal{A}^{-1}\mathcal{D}_i$ and $\mathcal{D}_i\mathcal{A}^{-1}$ we obtain

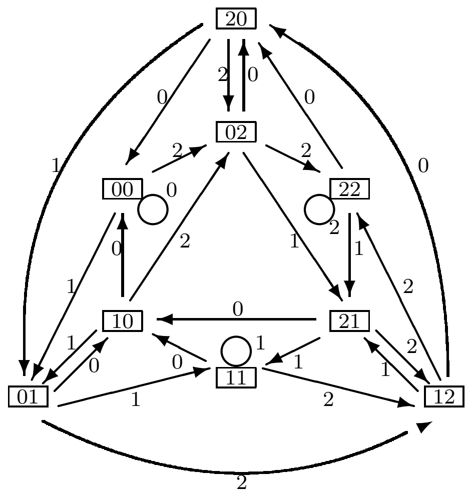
$$|\mathcal{A}^{-1}\mathcal{X}_i| \cdot q^{-n+1} + |\mathcal{A}^{-1}\mathcal{Y}_i| \cdot q^{-n} = Lq^{-n+1} + (q^n - qL)q^{-n} = 1 .$$

In a similar way we obtain $|\mathcal{X}_i\mathcal{A}^{-1}| \cdot q^{-n+1} + |\mathcal{Y}_i\mathcal{A}^{-1}| \cdot q^{-n} = 1$.

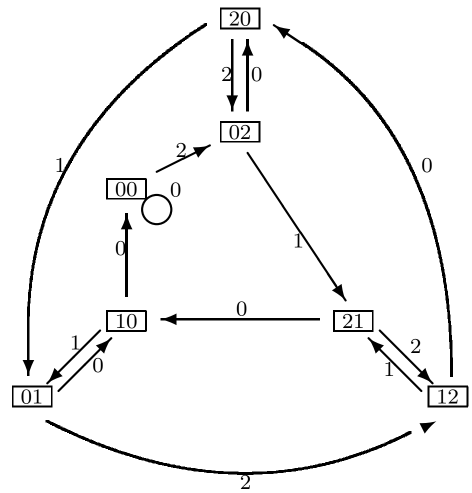
It follows, that $\mathcal{A}^{-1}\mathcal{D}_i$ is maximal prefix-free and $\mathcal{D}_i\mathcal{A}^{-1}$ is maximal suffix-free. This shows that $\mathcal{D}$ is a π-system with π-partition $\mathcal{D}_1, \ldots \mathcal{D}_k$. Especially, if $1 \leq L < q^{n-1}$ then $|\mathcal{Y}_i| = q^n - qL > 0$ and therefore $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ is a two level $\pi_q(n+1;k)$-system with
$|\mathcal{D} \cap \mathcal{A}^n| = |\mathcal{X}| = kL$. This shows part (ii) of Theorem 17 and moreover that any two level π-system $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ can be constructed as described in Construction 1. If $L = q^{n-1}$ then $\mathcal{Y} = \emptyset$ and $\mathcal{V} = \mathcal{A}^{n-1}$. Therefore $\mathcal{X}$ is the edge set of a k-factor in $\mathcal{B}_q(n-1)$ and $\mathcal{D} \subseteq \mathcal{A}^n$ is a one level $\pi_q(n;k)$-system. This shows part (iii) of Theorem 17. **q.e.d**

We give an example for Construction 1, by constructing a two level $\pi_3(4;2)$-system for $L = 7$. Let $\mathcal{A} = \{0,1,2\}$. We need a 2-regular subgraph $\Lambda := (\mathcal{V}, \mathcal{X}) \subseteq (\mathcal{A}^2, \mathcal{A}^3)$ in $\mathcal{B}_3(2)$ with $|\mathcal{V}| = 7$ and two edge disjoint 1-factors Λ_1, Λ_2 of Λ. With $X_1, \mathcal{X}_2$ we denote the edge sets of Λ_1 and Λ_2. The pictures below show such subgraphs in $\mathcal{B}_3(2)$ and their successor maps $\mathcal{F}, \mathcal{F}_1$ and $\mathcal{F}_2$.

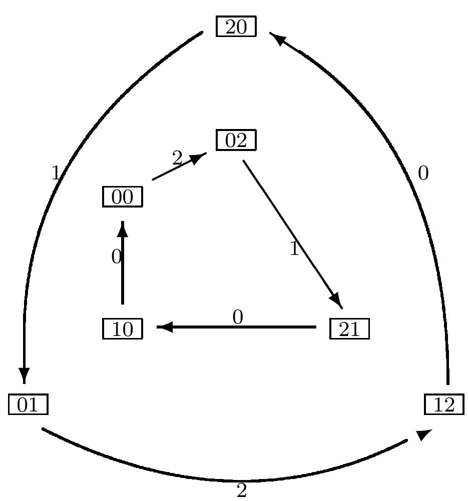
$\mathcal{B}_3(2)$

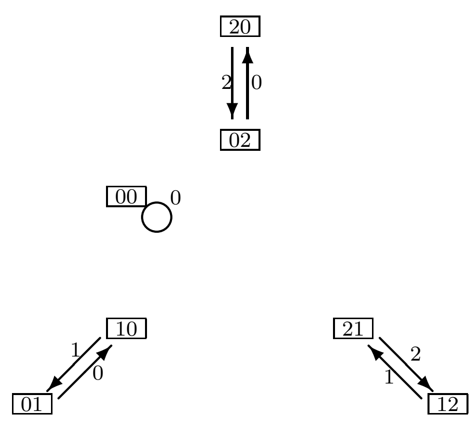


2-regular subgraph $\Lambda \subseteq \mathcal{B}_3(2)$



edge disjoint 1-factors Λ_1 and Λ_2 of Λ





Successor maps $\mathcal{F}, \mathcal{F}_1$ and $\mathcal{F}_2$ of the graphs Λ, Λ_1 and Λ_2:

$v \in \mathcal{A}^2$	$\mathcal{F}(v)$	$\mathcal{F}_1(v)$	$\mathcal{F}_2(v)$	$v \in \mathcal{A}^2$	$\mathcal{F}(v)$	$\mathcal{F}_1(v)$	$\mathcal{F}_2(v)$	$v \in \mathcal{A}^2$	$\mathcal{F}(v)$	$\mathcal{F}_1(v)$	$\mathcal{F}_2(v)$
00	$\{0,2\}$	2	0	01	$\{0,2\}$	2	0	02	$\{0,1\}$	1	0
10	$\{0,1\}$	0	1	11	$\emptyset$	$\emptyset$	$\emptyset$	12	$\{0,1\}$	0	1
20	$\{1,2\}$	1	2	21	$\{0,2\}$	0	2	22	$\emptyset$	$\emptyset$	$\emptyset$

For the vertex set $\mathcal{V}$, the set $\mathcal{V}^c$ and the edge sets $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2$ of Λ, Λ_1 and Λ_2 we obtain:

$$\begin{array}{lcl} \mathcal{V} & = & \{00, 01, 02, 10, 12, 20, 21\} \quad , \quad \mathcal{V}^c = \mathcal{A}^2 - \mathcal{V} = \{11, 22\} \ , \\ \mathcal{X}_1 & = & \{000, 010, 020, 101, 121, 202, 212\} \ , \\ \mathcal{X}_2 & = & \{002, 012, 021, 100, 120, 201, 210\} \ , \\ \mathcal{X} & = & \{000, 002, 010, 012, 020, 021, 100, 101, 120, 121, 201, 202, 210, 212\} \end{array}$$

We define the permutations $\varphi_1, \varphi_2 : \mathcal{A} \leftrightarrow \mathcal{A}$ as:

$a \in \mathcal{A}$	$\varphi_1(a)$	$\varphi_2(a)$
0	0	1
1	1	2
2	2	0

Obviously $\varphi_1(a) \neq \varphi_2(a)$ holds for all $a \in \mathcal{A}$. If we let $\mathcal{Y}_i := \bigcup_{a=0}^{2} a\mathcal{V}^c\varphi_i(a)$ for $i \in \{1, 2\}$ and $\mathcal{Y} := \mathcal{Y}_1 \cup \mathcal{Y}_2$ we obtain:

$$\begin{array}{lcl} \mathcal{Y}_1 & = & \{0110, 0220, 1111, 1221, 2112, 2222\} \\ \mathcal{Y}_2 & = & \{0111, 0221, 1112, 1222, 2110, 2220\} \\ \mathcal{Y} & = & \{0110, 0111, 0220, 0221, 1111, 1112, 1221, 1222, 2110, 2112, 2220, 2222\} \end{array}$$

Let $\mathcal{D} := \mathcal{X} \cup \mathcal{Y}$ and $\mathcal{D}_i := \mathcal{X}_i \cup \mathcal{Y}_i$ for $i \in \{1, 2\}$, then $\mathcal{D} \subseteq \mathcal{A}^3 \cup \mathcal{A}^4$ is a two level $\pi_3(4; 2)$-system with π-partition $\mathcal{D}_1, \mathcal{D}_2$ and $|\mathcal{D} \cap \mathcal{A}^3| = |\mathcal{X}| = 14 = 2L$. For the Kraftsum we obtain:

$$S(\mathcal{D}) = |\mathcal{X}| \cdot \frac{1}{3^3} + |\mathcal{Y}| \cdot \frac{1}{3^4} = \frac{14}{3^3} + \frac{12}{3^4} = \frac{54}{3^4} = \frac{2}{3}$$

Using Theorem 17 we can prove now Lemma 48. This was:

> Let $n \in \mathbb{N}$ and $|\mathcal{A}| = 2$. For any $\beta_n, \beta_{n+1} \in \mathbb{N}_0$ with $\frac{\beta_n}{2^n} + \frac{\beta_{n+1}}{2^{n+1}} = \frac{1}{2}$ there exists a $\pi_2(0, \ldots, 0, \beta_n, \beta_{n+1}\,;\, 1)$-system.

Proof of Lemma 48: Let $n \in \mathbb{N}$ and $\mathcal{A} = \{0, 1\}$. From $\frac{\beta_n}{2^n} + \frac{\beta_{n+1}}{2^{n+1}} = \frac{1}{2}$ follows, that $0 \leq \beta_n \leq 2^{n-1}$ and $\beta_{n+1} = 2^n - 2\beta_n$. For $\beta_n \neq 0$ follows from Lempel's Theorem 39, that there exists a cycle in $\mathcal{B}_2(n-1)$ of length β_n, i.e. there exists a 1-regular subgraph in $\mathcal{B}_2(n-1)$ with β_n vertices. By Theorem 17 (i) and (ii) it follows for $1 \leq \beta_n < 2^{n-1}$, that there exists a two level $\pi_2(n+1, 1)$-system $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ with $|\mathcal{D} \cap \mathcal{A}^n| = \beta_n$ and $|\mathcal{D} \cap \mathcal{A}^{n+1}| = 2^n - 2\beta_n = \beta_{n+1}$. Especially $\mathcal{D}$ is a $\pi_2(0, \ldots, 0, \beta_n, \beta_{n+1}\,;\, 1)$-system. If $\beta_n = 2^{n-1}$ then $\beta_{n+1} = 0$ and the cycle is a Hamilton circuit in $\mathcal{B}_2(n-1)$, i.e. a 1-factor of $\mathcal{B}_2(n-1)$. It follows from

Theorem 17 (iii), that the edge set of the cycle is a one-level $\pi_2(n;1)$-system, i.e. a $\pi_2(0,\ldots,0,\beta_n,\beta_{n+1}\,;\,1)$-system. Also for $\beta_{n+1} = 2^n$ and $\beta_n = 0$ there exists a $\pi_2(0,\ldots,0,\beta_n,\beta_{n+1}\,;\,1)$-system. This follows by the same argument,i.e the edge set of a Hamilton circuit in $\mathcal{B}_2(n)$ is a $\pi_2(0,\ldots,0,\beta_{n+1}\,;\,1)$-system. **q.e.d**

For $\mathcal{A} = \{0,1\}$ from Construction 1 follows, that we obtain a two level $\pi_2(n+1;1)$-system $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ with $|\mathcal{D} \cup \mathcal{A}^n| = L$, if we choose $\mathcal{D} \cap \mathcal{A}^n$ to be the edge set of a cycle in $\mathcal{B}_2(n-1)$ of length L and $\mathcal{D} \cap \mathcal{A}^{n+1} = 0\mathcal{V}^c 0 \cup 1\mathcal{V}^c 1$.[2] Furthermore every one level $\pi_2(n;1)$-system is the edge set of a Hamilton circuit in $\mathcal{B}_2(n-1)$ and vice versa.

¿From Theorem 17 and Theorem 16 we obtain the following generalization of Yekhanin's Theorem 15 for arbitrary alphabets:

Theorem 18 *Let* $|\mathcal{A}| = q \geq 2\,,\quad 1 \leq k < q$, $(\alpha_l)_{l\in\mathbb{N}}$ *be a sequence of nonnegative integers with* $\sum\limits_{l=1}^{\infty} \alpha_l q^{-l} \leq \gamma_k$ *,where* γ_k *is chosen as in Theorem 16, and* $n \in \mathbb{N}$ *be the first integer with* $\alpha_n \neq 0$*.*

(i) If $\frac{\alpha_n}{q^n} + \frac{\alpha_{n+1}}{q^{n+1}} \geq \frac{k}{q}$ *,* $\alpha_n = kL$ *for some* $1 \leq L < q^{n-1}$ *and there exists a* k*-regular subgraph in* $\mathcal{B}_q(n-1)$ *with* L *vertices, then there exists a fix-free code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

(ii) If $\frac{\alpha_n}{q^n} \geq \frac{k}{q}$ *then there exists a fix-free code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

Proof: Let $\mathcal{A}, k, (\alpha_l)_{l\in\mathbb{N}}$ and n be as in the theorem. Furthermore let $\frac{\alpha_n}{q^n} + \frac{\alpha_{n+1}}{q^{n+1}} \geq \frac{k}{q}$ and $\alpha_n = kL$ for some $1 \leq L < q^{n-1}$. Because of $\alpha_n < kq^{n-1}$ it follows, that there exists $1 \leq \beta \leq \alpha_{n+1}$ with

$$\frac{\alpha_n}{q^n} + \frac{\beta}{q^{n+1}} = \frac{k}{q}\,. \tag{4.50}$$

We obtain $\beta = k(q^n - qL)$.

Let us assume that there exists a k-regular subgraph with L vertices in $\mathcal{B}_q(n-1)$. Then from Theorem 17 (ii) and (i) follows, that there exists a two level $\pi_q(n+1;k)$-system $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ with $|\mathcal{D} \cap \mathcal{A}^n| = kL = \alpha_n$ and $|\mathcal{D} \cap \mathcal{A}^{n+1}| = k(q^n - qL) = \beta$. $\mathcal{D}$ is a $\pi_q(\alpha_1,\ldots\alpha_n,\beta;k)$-system. By (4.50) and Theorem 17 follows, that there exists a fix-free extension of $\mathcal{D}$ which fits to $(\alpha_l)_{l\in\mathbb{N}}$. This shows (i). Part (ii)

[2] $\mathcal{V}^c$ is the set of vertices, which do not lay on the cycle.

of the theorem has been shown already in Proposition 55. Another proof is the following:
If $\frac{\alpha_n}{q^n} \geq \frac{k}{q}$ then there exists $0 \leq \beta \leq \alpha_n$ such that $\frac{\beta}{q^n} = \frac{k}{q}$. Then $\beta = kq^n$. Moreover there exists a k-factor in $\mathcal{B}_q(n-1)$. Therefore from Theorem 17 (iii) follows, that there exists a one level $\pi_q(n;k)$-system $\mathcal{D} \subseteq \mathcal{A}^n$. Obviously $\mathcal{D}$ is a $\pi_q(\alpha_1, \ldots, \alpha_{n-1}, \beta; k)$-system and therefore from Theorem 16 follows that there exists a fix-free extension of $\mathcal{D}$ which fits to $(\alpha_l)_{l\in\mathbb{N}}$. **q.e.d**

As shown in the proof of Corollary 16, we have $\gamma_{\lceil \frac{q}{2} \rceil} \geq \frac{3}{4}$ for all $q \geq 2$, therefore we obtain for $k = \lceil \frac{q}{2} \rceil$ the following corollary of Theorem 18 :

Corollary 4 *Let $|\mathcal{A}| = q \geq 2$, $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers with $\sum\limits_{l=1}^{\infty} \alpha_l q^{-l} \leq \frac{3}{4}$. Let $n \in \mathbb{N}$ be the first integer with $\alpha_n \neq 0$.*

(i) If $\frac{\alpha_n}{q^n} + \frac{\alpha_{n+1}}{q^{n+1}} \geq \lceil \frac{q}{2} \rceil \frac{1}{q}$, $\alpha_n = \lceil \frac{q}{2} \rceil L$ for some $1 \leq L < q^{n-1}$ and there exists a $\lceil \frac{q}{2} \rceil$-regular subgraph in $\mathcal{B}_q(n-1)$ with L vertices then there exists a fix-free code which fits to $(\alpha_l)_{l\in\mathbb{N}}$.

(ii) If $\frac{\alpha_n}{q^n} \geq \lceil \frac{q}{2} \rceil \frac{1}{q}$ then there exists a fix-free code which fits to $(\alpha_l)_{l\in\mathbb{N}}$.

Let $\mathcal{A} = \{0, 1\}$. By Lempels Theorem 39 follows, that for every $1 \leq L \leq 2^{n-1}$ there exists a cycle of length L in $\mathcal{B}_2(n-1)$, i.e. there exists a 1-regular subgraph in $\mathcal{B}_2(n-1)$ with L vertices. Therefore we obtain from Corollary 4, Yekhanin's Theorem 15, which was:

> Let $|\mathcal{A}| = 2$,$(\alpha_l)_{n\in\mathbb{N}}$ be a sequence of nonnegative integers with $\sum\limits_{l=1}^{\infty} \alpha_l q^{-l} \leq \frac{3}{4}$. Let $n \in \mathbb{N}$ be the smallest integer with $\alpha_n \neq 0$.
> If $\frac{\alpha_n}{q^n} + \frac{\alpha_{n+1}}{q^{n+1}} \geq \frac{1}{2}$ then there exists a fix-free code which fits to $(\alpha_l)_{n\in\mathbb{N}}$.

In the generalization of Theorem 15 for arbitrary alphabets, Theorem 18 and Corollary 4, two extra conditions occur. First $\alpha_n = kL$ for some $1 \leq L \leq q^{n-1}$, if $\alpha_n \leq kq^{n-1}$ and secondly there has to exists a k-regular subgraph in $\mathcal{B}_q(n-1)$ with L vertices. One can ask if there is a generalization of Theorem 15, without such extra conditions ? However, if we take into account, that Theorem 17 gives us a one-to-one correspondence between two level π-systems $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ and regular subgraphs in de Bruin digraphs, it is obviously, that Theorem 18 and Corollary 4 are the best generalizations of Yekhanin's original Theorem 15, which can be obtained by using the technique of π-systems. One can only try to replace the condition for the existence of regular subgraphs in Theorem 18 and Corollary 4

by the values of L, for which k-regular subgraphs with L vertices in $\mathcal{B}_q(n-1)$ exist. It was shown in Chapter 3 Theorem 14, there do not exist k-regular subgraphs with L vertices in $\mathcal{B}_q(n-1)$, if $L < k^{n-1}$ or $k^{n-1} < L < k^{n-1} + k^{n-2}$. This means, that for these values of L there does not exist a $\pi_q(n;k)$-system $\mathcal{D} \subseteq \mathcal{A}^n \cup \mathcal{A}^{n+1}$ with kL codewords on the n-th level. Furthermore in Chapter 3 there are several constructions of k-regular subgraphs of $\mathcal{B}_q(n)$ for certain values of L.

Chapter 5

The $\frac{3}{4}$-conjecture for binary fix-free codes

In this chapter we examine the $\frac{3}{4}$-conjecture for the special case $|\mathcal{A}| = 2$. There are some results which was shown only for this case.

In [10] Kukorelly and Zeger have shown the following theorem.

Theorem 19 (Kukorelly and Zeger [10]) *Let $|\mathcal{A}| = 2$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{N}_0$. If $\sum_{l=1}^{n} \alpha_l \left(\frac{1}{2}\right)^l \leq \frac{3}{4}$ and $\alpha_l \leq 2$ for all $1 \leq l \leq n$, then there exists a fix-free set $\mathcal{C} \subseteq \mathcal{A}^*$ which fits to $(\alpha_1, \ldots, \alpha_n)$.*

To prove the theorem, Kukorelly and Zeger distinguish eight cases, where the theorem is easy to show or follows from other theorems for the first seven cases. We show the theorem only for this seven easy cases, a proof of the theorem for the last case can be found in [10].

Proof: Let $|\mathcal{A}| = 2$ and $(\alpha_1, \ldots, \alpha_n)$ as in the theorem. It is sufficient to show that the theorem holds for all $(\alpha_1, \ldots m, \alpha_n)$ with Kraftsum $\frac{3}{4}$. We distinguish eight cases:

Case 1: $\alpha_1 = 1$
In this case the theorem follows from Theorem 15.

Case 2: $\alpha_1 = 0$ and $\alpha_2 = 2$
Also in this case the theorem follows from Theorem 15.

Case 3: $\alpha_1 = \alpha_2 = 0$
In this case with theorem 6 follows that the Theorem holds.

Case 4: $\alpha_1 = 0\,, \quad \alpha_2 = 1$ and $\alpha_3 \geq 2$
In this case the theorem follows from Theorem 15.

Case 5: $\alpha_1 = 0\,, \quad \alpha_2 = 1\,, \quad \alpha_3 \leq 1$ and $n = 3$
In this case the theorem follows from Theorem 5.

Case 6: $\alpha_1 = 0\,, \quad \alpha_2 = 1\,, \quad \alpha_3 \leq 1$ and $n = 4$
While the Kraftsum of $(\alpha_1, \ldots, \alpha_4)$ is $\frac{3}{4}$ it follows, that either $\alpha_4 = 6$ or $\alpha_4 = 8$. Two examples for such fix-free codes are listed below:

$(\alpha_1, \ldots, \alpha_4)$ $= (0\,,1\,,1\,,6)$	$(\alpha_1, \ldots, \alpha_4)$ $= (0\,,1\,,0\,,8)$
11	11
101	0000
0000	0010
0010	0100
0100	0110
0110	1001
1001	1000
1000	1010
	0101

Case 7: $\alpha_1 = 0\,, \quad \alpha_2 = 1\,, \quad \alpha_3 \leq 1$ and $n = 5$
In this case there are six possibilities for $(\alpha_1, \ldots, \alpha_5)$. For each of them examples for fix-free codes are shown in the tabular below:

$(\alpha_1, \ldots, \alpha_5) =$ $(0\,,1\,,1\,,2\,,8)$	$(\alpha_1, \ldots, \alpha_5) =$ $(0\,,1\,,1\,,1\,,10)$	$(\alpha_1, \ldots, \alpha_5) =$ $(0\,,1\,,1\,,0\,,12)$	$(\alpha_1, \ldots, \alpha_5) =$ $(0\,,1\,,0\,,2\,,12)$	$(\alpha_1, \ldots, \alpha_5) =$ $(0\,,1\,,0\,,1\,,14)$	$(\alpha_1, \ldots, \alpha_5) =$ $(0\,,1\,,1\,,0\,,16)$
11	11	11	11	11	11
101	101	101	1001	1001	00000
1001	1001	00000	0110	00000	00010
0110	00000	00010	00000	00010	00100
00000	00010	00100	00010	00100	00110
00010	00100	00110	00100	00110	01000
00100	00110	01000	01000	01000	01010
01000	01000	01010	01010	01010	01100
01010	01010	01100	01110	01100	01110
01110	01100	01110	10001	01110	10001
10001	01110	10001	10000	10001	10000
10000	10001	10000	00001	10000	00001
	10000	10010	00101	00001	00101
		01001	10100	00101	10100
			10101	10100	10101
				10101	10010
					01001

Case 8: $\alpha_1 = 0\,, \quad \alpha_2 = 1\,, \quad \alpha_3 \leq 1$ and $n \geq 6$
For this case a proof of the theorem can be found in [10]. **q.e.d**

Ye and Yeung have shown in [7] some results which are related to the binary $\frac{3}{4}$-conjecture. Especially they prove a sufficient and a necessary condition for the existence of binary fix-free codes, where the conditions depends on the lengths sequence of a fix-free code. Let us remind that a lengths sequence $\vec{l_n} = (l_1, \ldots, l_n)$ is an increasing finite sequence of natural numbers. A code $\mathcal{C}$ fits to the lengths sequence $\vec{l_n}$, if the numbers $l_1, \ldots, l_n$ are the lengths of the codewords in $\mathcal{C}$.

We define for a number $x \in \mathbb{R}$:

$$x^+ := \begin{cases} x & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases} .$$

Let $\vec{l_n} = (l_1, \ldots, l_n) \in \mathbb{N}^n$ be a lengths sequence. We define $su(\vec{l_n})$, $ne(\vec{l_n})$ and $h(i)$ as:

$$\begin{aligned}
h(i) \quad &:= \quad \min\{j \,|\, l_j = l_{i+1}\} \quad \text{for all } 1 \le i < n\,, \\
su(\vec{l_n}) \quad &:= \quad \prod_{i=1}^{n-1} \Big(1 - 2 \sum_{1 \le j \le i} 2^{-l_i} + (\, i + 1 - h(i)\,) \cdot 2^{-l_i+1} + \sum_{\substack{1 \le j,k \le h(i)-1 \\ \text{s.t. } l_j + l_k \le l_i + 1}} 2^{-l_j - l_k}\Big)^+ \,, \\
ne(\vec{l_n}) \quad &:= \quad \prod_{i=1}^{n-1} \Big(1 - 2 \sum_{1 \le j \le i} 2^{-l_i} + (\, i + 1 - h(i)\,) \cdot 2^{-l_i+1} + \sum_{1 \le j,k \le h(i)-1} 2^{(l_{i+1} - l_j - l_k)^+ - l_{i+1}}\Big)^+ \,.
\end{aligned}$$

Theorem 20 (Ye and Yeung) *Let $|\mathcal{A}| = 2$ and $\vec{l_n} \in \mathbb{N}^n$ be a lengths sequence.*

(i) (Sufficient Condition) If $su(\vec{l_n}) > 0$, then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits to $\vec{l_n}$.

(ii) (Necessary Condition) If $ne(\vec{l_n}) = 0$, then there does not exist a fix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits to $\vec{l_n}$.

Furthermore Ye and Yeung have shown in [7] the following corollary of part (i) of the theorem above.

Corollary 5 (Ye and Yeung) *Let $|\mathcal{A}| = 2$ and $\vec{l_n} \in \mathbb{N}^n$ be a lengths sequence. If*

$$\sum_{1 \le j \le n} 2^{-l_i} < \frac{1}{2} + \frac{n + 2 - h(n-1)}{2} \cdot 2^{-l_n}\,,$$

then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits to $\vec{l_n}$.

Proofs of Theorem 20 and Corollary 5 can be found in [7].

Moreover Ye and Yeung have shown the following proposition.

Proposition 61 (Ye and Yeung) *Let $|\mathcal{A}| = 2$ and $\alpha_1, \dots, \alpha_n \in \mathbb{N}_0$. If $\alpha_1 = 1$ and $\sum_{l=1}^{n} \alpha_l \left(\frac{1}{2}\right)^l \leq \frac{5}{8}$, then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits to $(\alpha_1, \dots, \alpha_n)$.*

Ye and Yeung gave in [7] two different proofs of the proposition above. The first proof works with Theorem 20 and the second proof use Lemma 19. With a proof of Yekhanin we will show in the last section of this chapter, that the proposition also holds for sequences with $\alpha_1 = 0$. However the proposition above follows also from Theorem 15. If $(\alpha_l)_{l\in\mathbb{N}}$ is a sequence of nonnegative integers with $\alpha_1 = 1$, then $\frac{\alpha_1}{2} + \frac{\alpha_2}{2^2} \geq \frac{1}{2}$. Therefore we obtain by Theorem 15 the more general proposition:

Proposition 62 (Yekhanin) *Let $|\mathcal{A}| = 2$ and $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence with $\sum_{l=1}^{n} \alpha_l \left(\frac{1}{2}\right)^l \leq \frac{3}{4}$. If $\alpha_1 = 1$ then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits to $(\alpha_l)_{l\in\mathbb{N}}$.*

The binary $\frac{3}{4}$-conjecture was verified by computer research for several sequences. The results are collected in the proposition below.

Proposition 63 *Let $\mathcal{A} = \{0, 1\}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{N}_0$ with $\sum_{l=1}^{n} \alpha_l 2^{-l} \leq \frac{3}{4}$.*

(i) **Ye and Yeung [7]**
If $n < 8$, then there exists a fix-free set $\mathcal{C} \subseteq \mathcal{A}^*$ which fits to $(\alpha_1, \dots, \alpha_n)$.

(ii) **Yekhanin [8]**
If $n < 9$, then there exists a fix-free set $\mathcal{C} \subseteq \mathcal{A}^*$ which fits to $(\alpha_1, \dots, \alpha_n)$.

5.1 Binary fix-free codes obtained from quaternary fix-free codes

In Chapter 2 and Chapter 4, we gave a lot of results for the q-ary case of the $\frac{3}{4}$-conjecture. By identifying the letters in $\{0,1,2,3\}$ with the the words of length 2 in $\{0,1\}^2$, it is possible to obtain some new results for the binary $\frac{3}{4}$-conjecture from the old 4-ary results in Chapter 2 and Chapter 4.

In this section we denote with $\mathcal{A}$ and $\mathcal{B}$ the alphabets $\mathcal{A} := \{0,1\}$ and $\mathcal{B} := \{0,1,2,3\}$. Let $\phi : \mathcal{B} \leftrightarrow \mathcal{A}^2$ be a bijection. For example:

$$\phi(0) = 00, \phi(1) = 01, \phi(2) = 10 \text{ and } \phi(3) = 11\,.$$

Let $w = w_1 \dots w_{2n} \in \mathcal{A}^{2n}, v = v_1 \dots v_n \in \mathcal{B}^n$ with $w_1, \dots, w_{2n} \in \mathcal{A}, v_1, \dots, v_n \in \mathcal{B}$ and $\mathcal{C} \subseteq \bigcup\limits_{l=1}^{\infty} \mathcal{A}^{2l}$, $\mathcal{D} \subseteq \mathcal{B}^+$. We define $\phi(v), \phi^{-1}(w), \phi(\mathcal{D})$ and $\phi^{-1}(\mathcal{C})$ as follows:

$$\begin{array}{lcl}
\phi(v) & := & \phi(v_1)\dots\phi(v_n) \in \mathcal{A}^{2n}\,, \\
\phi^{-1}(w) & := & \phi^{-1}(w_1w_2)\phi^{-1}(w_3w_4)\dots\phi^{-1}(w_{2n-1}w_{2n}) \in \mathcal{B}^n\,, \\
\phi(\mathcal{C}) & := & \{\phi(v) \in \mathcal{A}^+ \mid v \in \mathcal{D}\} \subseteq \mathcal{B}^+\,, \\
\phi^{-1}(\mathcal{D}) & := & \{\phi^{-1}(w) \mid w \in \mathcal{D}\} \subseteq \bigcup\limits_{l=1}^{\infty} \mathcal{A}^{2l}\,, \\
\phi(e) & := & e \text{ and } \phi^{-1}(e) := e\,.
\end{array}$$

Obviously the map ϕ is a one-to-one map from $\mathcal{B}^*$ onto $\bigcup\limits_{l=0}^{\infty} \mathcal{A}^{2l}$ with inverse map ϕ^{-1}. Furthermore we obtain:

$$\phi(\mathcal{B}^n) = \mathcal{A}^{2n} \text{ for all } n \in \mathbb{N}_0\,. \tag{5.1}$$

It is easy to verify, that the following equations hold:

$$\begin{array}{lcll}
\phi(uv) & = & \phi(u)\phi(v) & \text{for all } u, v \in \mathcal{B}^*\,, \\
\phi^{-1}(u'v') & = & \phi^{-1}(u)\phi^{-1}(v) & \text{for all } u', v' \in \bigcup\limits_{l=0}^{\infty} \mathcal{A}^{2l}\,.
\end{array} \tag{5.2}$$

Lemma 64 *Let $\mathcal{A} = \{0,1\}, \mathcal{B} = \{0,1,2,3\}$, $\phi : \mathcal{B} \leftrightarrow \mathcal{A}^2$ be a bijection and $\mathcal{C} \subseteq \mathcal{A}^+, \mathcal{D} \subseteq \mathcal{B}^+$ such that $\phi(\mathcal{D}) = \mathcal{C}$.*

(i) $|\mathcal{C} \cap \mathcal{A}^{2l+1}| = 0$ and $|\mathcal{C} \cap \mathcal{A}^{2l}| = |\mathcal{D} \cap \mathcal{B}^l|$ for all $l \in \mathbb{N}_0$.

(ii) $\mathcal{C}$ is fix-free if and only if $\mathcal{D}$ is fix-free.

(iii) $S(\mathcal{C}) = \sum\limits_{l=0}^{\infty} |\mathcal{C} \cap \mathcal{A}^l| \left(\frac{1}{2}\right)^l = \sum\limits_{l=0}^{\infty} |\mathcal{D} \cap \mathcal{B}^l| \left(\frac{1}{4}\right)^l = S(\mathcal{D})$

Proof:

(i) $|\mathcal{C} \cap \mathcal{A}^{2l+1}| = 0$ for all $l \in \mathbb{N}_0$, because $\phi(\mathcal{D}) = \mathcal{C}$ and $\phi : \mathcal{B} \leftrightarrow \bigcup_{l=0}^{\infty} \mathcal{A}^{2l}$ is a bijection. Furthermore we have $\phi(B^l) = \mathcal{A}^{2l}$ for all $l \in \mathbb{N}_0$. Therefore we obtain for all $l \in \mathbb{N}_0$:

$$|\mathcal{C} \cap \mathcal{A}^{2l}| = |\phi(\mathcal{D}) \cap \mathcal{A}^{2l}| = |\phi(\mathcal{D} \cap \mathcal{B}^l)| = |\mathcal{D} \cap \mathcal{B}^l| \,.$$

(ii) This follows by (5.2).

(iii) Since $\phi_{|\mathcal{B}^l} : \mathcal{B}^l \leftrightarrow \mathcal{A}^{2l}$ is a bijection and $\phi(\mathcal{D}) = \mathcal{C} \subseteq \bigcup_{l=0}^{\infty} \mathcal{A}^{2l}$, we obtain:

$$\begin{array}{rclcl} S(\mathcal{C}) & = & \sum_{l=0}^{\infty} |\mathcal{C} \cap \mathcal{A}^l| \left(\frac{1}{2}\right)^l & = & \sum_{l=0}^{\infty} |\mathcal{C} \cap \mathcal{A}^{2l}| \left(\frac{1}{2}\right)^{2l} \\ & = & \sum_{l=0}^{\infty} |\phi(\mathcal{D}) \cap \mathcal{A}^{2l}| \left(\frac{1}{2}\right)^{2l} & = & \sum_{l=0}^{\infty} |\phi(\mathcal{D} \cap \mathcal{B}^l)| \left(\frac{1}{2^2}\right)^{l} \\ & = & \sum_{l=0}^{\infty} |\mathcal{D} \cap \mathcal{B}^l| \left(\frac{1}{4}\right)^l & = & S(\mathcal{D}) \,. \quad \textbf{q.e.d} \end{array}$$

If we use the above lemma together with the theorems for the q-ary case in Chapter 2 and Chapter 4, we obtain the following proposition.

Proposition 65 *Let $\mathcal{A} := \{0, 1\}$ and $(\alpha_l)_{l \in \mathbb{N}}$ be a sequence of nonnegative integers with $\sum_{l=1}^{\infty} \alpha_l \left(\frac{1}{2}\right)^l \leq \frac{3}{4}$.*

(i) If there exists an $n \geq 2$ such that $\alpha_2 = \alpha_{2l+1} = 0$ for all $l \in \mathbb{N}_0$, $\alpha_{2l} = 2^l$ for all $2 \leq l < n$, $\alpha_{2n} \geq 2^{n+1}$ and $\alpha_{2l} \in \mathbb{N}_0$ for all $l > n$, then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits to $(\alpha_l)_{l \in \mathbb{N}}$.

(ii) If there exists an $n \geq 3$ such that $\alpha_2 = \alpha_4 = \alpha_{2l+1} = 0$ for all $l \in \mathbb{N}_0$, $\alpha_{2l} = 2^{l+1}$ for all $2 \leq l < n$, $\alpha_{2n} \geq 2^{n+2}$ and $\alpha_{2l} \in \mathbb{N}_0$ for all $l > n$, then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits to $(\alpha_l)_{l \in \mathbb{N}}$.

(iii) If there exists an $n \in \mathbb{N}$ such that $\alpha_2 = \alpha_4 = \ldots = \alpha_{2n-2} = \alpha_{2l+1} = 0$ for all $l \in \mathbb{N}_0$, α_{2n} is even, $\frac{\alpha_{2n}}{2^{2n}} + \frac{\alpha_{2n+2}}{2^{2n+2}} \geq \frac{1}{2}$ and there exists a 2-regular subgraph of $\mathcal{B}_4(n-1)$ with $\frac{\alpha_{2n}}{2}$ vertices, then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits to $(\alpha_l)_{l \in \mathbb{N}}$.

(iv) If there exists an $n \in \mathbb{N}$ such that $\alpha_2 = \alpha_4 = \ldots = \alpha_{2n-2} = \alpha_{2l+1} = 0$ for all $l \in \mathbb{N}_0$ and $\frac{\alpha_{2n}}{2^{2n}} \geq \frac{1}{2}$, then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits to $(\alpha_l)_{l \in \mathbb{N}}$.

(v) Let $l_{min} := \min\{l \,|\, \alpha_l \neq 0\}$ and $l_{max} := \sup\{l \,|\, \alpha_l \neq 0\}$. If $l_{max} < \infty$, $4 \leq l_{min}$ is even, $\alpha_{2l+1} = 0$ for all $l \in \mathbb{N}_0$ and $\alpha_{2l} \leq 2^{\frac{l_{min}}{2}-2+l}$ for all $2l \neq l_{max}$, then there exists a fix-free code $\mathcal{C} \subseteq \mathcal{A}^+$ which fits to $(\alpha_l)_{l\in\mathbb{N}}$.

Proof: Let $\mathcal{B} := \{0,1,2,3\}$ and $\phi : \mathcal{B} \leftrightarrow \mathcal{A}^2$ be a bijection. We define the sequence $(\beta_l)_{l\in\mathbb{N}}$ as:

$$\beta_l := \alpha_{2l} \quad \text{for all } \ l \in \mathbb{N}\,.$$

In all cases of the proposition we have $\alpha_{2l+1} = 0$ for all $l \in \mathbb{N}$. Let us assume that $\mathcal{D} \subseteq \mathcal{B}^+$ is a fix-free code which fits to $(\beta_l)_{l\in\mathbb{N}}$. By Lemma 64 follows, that $\mathcal{C} := \phi(\mathcal{D}) \subseteq \mathcal{A}^+$ is a fix-free code which fits to $(\alpha_l)_{l\in\mathbb{N}}$. Therefore it is for all cases of the proposition sufficient to show that there exists a fix-free code $\mathcal{D} \subseteq \mathcal{B}^+$ which fits to $(\beta_l)_{l\in\mathbb{N}}$. We obtain for the Kraftsum of $(\beta_l)_{l\in\mathbb{N}}$:

$$\sum_{l=0}^{\infty} \beta_l \left(\frac{1}{4}\right)^l = \sum_{l=0}^{\infty} \alpha_{2l} \left(\frac{1}{2}\right)^{2l} = \sum_{l=0}^{\infty} \alpha_l \left(\frac{1}{2}\right)^l \leq \frac{3}{4}\,.$$

(i) In this case we obtain for $(\beta_l)_{l\in\mathbb{N}}$:

$$\beta_1 = 0, \beta_l = 2^l = \left(\frac{4}{2}\right)^l \quad \text{for all } \ 2 \leq l < n \ \text{ and } \ \beta_n \geq 2^{n+1} = 4\left(\frac{4}{2}\right)^{n-1}\,.$$

By Proposition 53 follows, that there exist a fix-free code $\mathcal{D} \subseteq \mathcal{B}^+$ which fits to $(\beta_l)_{l\in\mathbb{N}}$.

(ii) In this case from Proposition 58 follows that there exists a fix-free code $\mathcal{D} \subseteq \mathcal{B}^+$ which fits to $(\beta_l)_{l\in\mathbb{N}}$.

(iii) In this case from Corollary 4.37 (i) follows that there exists a fix-free code $\mathcal{D} \subseteq \mathcal{B}^+$ which fits to $(\beta_l)_{l\in\mathbb{N}}$.

(iv) In this case from Corollary 4.37 (ii) follows that there exists a fix-free code $\mathcal{D} \subseteq \mathcal{B}^+$ which fits to $(\beta_l)_{l\in\mathbb{N}}$.

(v) Let $l'_{min} := \min\{l \,|\, \beta_l \neq 0\}$ and $l'_{max} := \sup\{l \,|\, \beta_l \neq 0\}$. It follows that $l'_{max} = \frac{l_{max}}{2} < \infty$ and $l'_{min} = \frac{l_{min}}{2} \geq 2$. Furthermore we obtain:

$$\beta_l = \alpha_{2l} \leq 2^{l'_{min}-2+l} = 4^{l'_{min}-2} \cdot 2^2 \cdot 2^{l-l'_{min}} \quad \text{for all } \ l \neq l_{max}\,.$$

By Theorem 7 follows, that there exists a fix-free code $\mathcal{D} \subseteq \mathcal{B}^+$ which fits to $(\beta_l)_{l\in\mathbb{N}}$. **q.e.d**

5.2 Binary fix-free codes with Kraftsum $\frac{5}{8}$

For $\mathcal{C} \subseteq \mathcal{A}^*$ and $a, b \in \mathcal{A}$ we define:

$$\begin{aligned} {}^a\mathcal{C} &:= \{aw \in \mathcal{A}^* \mid aw \in \mathcal{C}\} = a\mathcal{A}^* \cap \mathcal{C} \\ \mathcal{C}^b &:= \{wb \in \mathcal{A}^* \mid wb \in \mathcal{C}\} = \mathcal{A}^* b \cap \mathcal{C} \\ {}^a\mathcal{C}^b &:= {}^a\mathcal{C} \cap \mathcal{C}^b = a\mathcal{A}^* b \cap \mathcal{C} = \{awb \in \mathcal{A}^* \mid awb \in \mathcal{C}\} \end{aligned}$$

We show first the following proposition:

Proposition 66 *Let* $|\mathcal{A}| = q\,,\ a, b \in \mathcal{A}\,,\ n \in \mathbb{N}$ *and* $\mathcal{C} \subseteq \bigcup_{l=1}^{n} \mathcal{A}^l$ *be fix-free then:*

(i)

$$|a\mathcal{A}^{n-1}b - \Delta_B^{n+1}(\mathcal{C})| \geq \max\left\{0, q^{n-1} - |\Delta_P^n({}^a\mathcal{C})| - |\Delta_S^n(\mathcal{C}^b)|\right\}\,,$$

(ii)

$$\begin{aligned} |\Delta_P^n({}^a\mathcal{C})| &= q^n \sum_{c \in \mathcal{A}} S({}^a\mathcal{C}^c)\,, \\ |\Delta_S^n(\mathcal{C}^b)| &= q^n \sum_{c \in \mathcal{A}} S({}^c\mathcal{C}^b)\,. \end{aligned}$$

Proof:
We show (i):

$$\begin{aligned} |a\mathcal{A}^{n-1}b - \Delta_B^{n+1}(\mathcal{C})| &= |(a\mathcal{A}^n \cap \mathcal{A}^n b) - (\Delta_P^{n+1}(\mathcal{C}) \cup \Delta_S^{n+1}(\mathcal{C}))| \\ &= |(a\mathcal{A}^n - \Delta_P^{n+1}(\mathcal{C})) \cap (\mathcal{A}^n b - \Delta_S^{n+1}(\mathcal{C}))| \\ &= |(a\mathcal{A}^{n-1} - \Delta_P^n(\mathcal{C}))\mathcal{A} \cap \mathcal{A}(\mathcal{A}^{n-1}b - \Delta_S^n(\mathcal{C}))| \\ \text{(with lemma 51 (ii))} \quad &\geq |a\mathcal{A}^{n-1} - \Delta_P^n(\mathcal{C})| + |\mathcal{A}^{n-1}b - \Delta_S^n(\mathcal{C})| - q^{n-1} \\ &= |a\mathcal{A}^{n-1} - \Delta_P^n({}^a\mathcal{C})| + |\mathcal{A}^{n-1} - \Delta_S^n(\mathcal{C}^b)| - q^{n-1} \\ &= q^{n-1} - |\Delta_P^n({}^a\mathcal{C})| + q^{n-1} - |\Delta_S^n(\mathcal{C}^b)| - q^{n-1} \\ &= q^{n-1} - |\Delta_P^n({}^a\mathcal{C})| - |\Delta_S^n(\mathcal{C}^b)|\,. \end{aligned}$$

We show (ii):

$$\begin{aligned} |\Delta_P^n({}^a\mathcal{C})| &= |\Delta_P^n(\bigcup_{c \in \mathcal{A}} {}^a\mathcal{C}^c)| = \sum_{l=1}^{n} |\mathcal{A}^l \cap \bigcup_{c \in \mathcal{A}} {}^a\mathcal{C}^c)| \cdot q^{n-l} \\ &= q^n \sum_{c \in \mathcal{A}} \sum_{l=1}^{n} |{}^a\mathcal{C}^c \cap \mathcal{A}^l| \cdot q^{-l} = q^n \sum_{c \in \mathcal{A}} S({}^a\mathcal{C}^c)\,. \end{aligned}$$

The second part of (ii) follows the same way. **q.e.d**

The next theorem was shown by Yekhanin in [9].

Theorem 21 (Yekhanin [9]) *Let $|\mathcal{A}| = 2$ and $(\alpha_l)_{l\in\mathbb{N}}$ a sequence of nonnegative integers with $\sum_{l=1}^{\infty} \alpha_l (\frac{1}{2})^l \leq \frac{5}{8}$ then there exists a fix-free $\mathcal{C} \subseteq \mathcal{A}^*$ which fits $(\alpha_l)_{l\in\mathbb{N}}$.*

Proof: Every sequence $(\alpha_l)_{l\in\mathbb{N}}$, with Kraftsum smaller than $\frac{5}{8}$ can be extended to a sequence $(\alpha'_l)_{l\in\mathbb{N}}$ with $\alpha'_l \geq \alpha_l$ for all $l \in \mathbb{N}$ and Kraftsum equal to $\frac{5}{8}$. Therefore it is sufficient to show the theorem for a sequence $(\alpha_l)_{l\in\mathbb{N}}$ of nonnegative integers with $\sum_{l=1}^{\infty} \alpha_l (\frac{1}{2})^l = \frac{5}{8}$.

We distinguish three cases:

Case 1: α_1=1
Then $\frac{\alpha_1}{2} = \frac{1}{2}$ and by Theorem 15 it follows, that there exist a fix-free code which fits to $(\alpha_l)_{l\in\mathbb{N}}$.

Case 2: $\alpha_1 = 0$ and $\alpha_2 = 2$
In this case $\frac{\alpha_2}{2^2} = \frac{1}{2}$ and by Theorem 15, there exists a fix-free code which fits to $(\alpha_l)_{l\in\mathbb{N}}$.

Case 3: $\alpha_1 = 0$ and $\alpha_2 < 2$
In this case we can find unique sequences of nonnegative integers $(\beta_l^{00})_{l\in\mathbb{N}}$, $(\beta_l^{01})_{l\in\mathbb{N}}$, $(\beta_l^{10})_{l\in\mathbb{N}}$ and $(\beta_l^{11})_{l\in\mathbb{N}}$ such that:

$$\begin{array}{l}
\sum_{l=1}^{\infty} \beta_l^{00} (\frac{1}{2})^l = \frac{1}{4}\,, \\
\sum_{l=1}^{\infty} \beta_l^{01} (\frac{1}{2})^l = \sum_{l=1}^{\infty} \beta_l^{10} (\frac{1}{2})^l = \sum_{l=1}^{\infty} \beta_l^{11} (\frac{1}{2})^l = \frac{1}{8}\,, \\
\alpha_l = \beta_l^{00} + \beta_l^{01} + \beta_l^{10} + \beta_l^{11} \quad \forall\, l \in \mathbb{N}\,, \\
\beta_m^{00} > 0 \Rightarrow \beta_l^{01} = \beta_l^{10} = \beta_l^{11} = 0 \quad \forall\, l \in \{1, \ldots, m-1\}\,, \\
\beta_m^{01} > 0 \Rightarrow \beta_l^{10} = \beta_l^{11} = 0 \quad \forall\, l \in \{1, \ldots, m-1\}\,, \\
\beta_m^{10} > 0 \Rightarrow \beta_l^{11} = 0 \quad \forall\, l \in \{1, \ldots, m-1\}\,.
\end{array}$$

To give an example, let the first eight terms of $(\alpha_l)_{l\in\mathbb{N}}, (\beta_l^{00})_{l\in\mathbb{N}}, \ldots, (\beta_l^{11})_{l\in\mathbb{N}}$ given by:

$$\begin{array}{ll}
(\alpha_1,\ldots,\alpha_8) & =(0\,,0\,,1\,,1\,,5\,,2\,,14\,,36\,,) \\
(\beta_1^{00},\ldots,\beta_8^{00}) & =(0\,,0\,,1\,,1\,,2\,,0\,,\ 0\ ,\ 0\,,) \\
(\beta_1^{01},\ldots,\beta_8^{01}) & =(0\,,0\,,0\,,0\,,3\,,2\,,\ 0\ ,\ 0\,,) \\
(\beta_1^{10},\ldots,\beta_8^{10}) & =(0\,,0\,,0\,,0\,,0\,,0\,,14\,,\ 4\,,) \\
(\beta_1^{11},\ldots,\beta_8^{11}) & =(0\,,0\,,0\,,0\,,0\,,0\,,\ 0\ ,32\,.)
\end{array}$$

and $\alpha_l = \beta_l^{00} = \ldots = \beta_l^{11} = 0$ for $l > 8$. Then $\sum\limits_{l=1}^{\infty} \alpha_l \cdot (\frac{1}{2})^l = \frac{5}{8}$ and the $(\beta_l^{ab})_{l\in\mathbb{N}}$ are the unique sequences with the above properties.

We construct by induction a fix-free $\mathcal{C} \subseteq \mathcal{A}^*$ such that $|\,{}^a\mathcal{C}^b \cap \mathcal{A}^l| = \beta_l^{ab}$ for all $l \in \mathbb{N}$, $a, b \in \{0,1\}$. $\mathcal{C}$ is a fix-free code which fits to $(\alpha_l)_{l\in\mathbb{N}}$, because $\mathcal{C}$ is the disjoint union of ${}^0\mathcal{C}^0$, ${}^1\mathcal{C}^0$, ${}^0\mathcal{C}^1$ and ${}^1\mathcal{C}^1$.
To construct a $\mathcal{C}$ with the above properties it is sufficient to find a sequence $\mathcal{C}_1 \subseteq C_2 \subseteq \mathcal{C}_3 \subseteq \ldots$ of fix-free sets such that $\mathcal{C}_n \subseteq \bigcup\limits_{l=1}^{n} \mathcal{A}^l$ and $|\,{}^a\mathcal{C}_n^b \cap \mathcal{A}^l| = \beta_l^{ab}$ for all $l \in \{1\ldots,n\}$, $a, b \in \{0,1\}$. Then we obtain $\mathcal{C}$ by $\mathcal{C} := \bigcup\limits_{l=1}^{\infty} \mathcal{C}_n$.
Let $\mathcal{C}_1 := \emptyset$, then $\mathcal{C}_1$ is fix-free and $|\,{}^a\mathcal{C}^b \cap \mathcal{A}^l| = \beta_1^{ab} = 0$ for all $a, b \in \{0,1\}$.

Let $\mathcal{C}_n \subseteq \bigcup\limits_{l=1}^{n} \mathcal{A}^l$ be a fix-free set such that $|\,{}^a\mathcal{C}_n^b \cap \mathcal{A}^l| = \beta_l^{ab}$ for all $l \in \{1,\ldots,n\}$, $a, b \in \{0,1\}$. Then we obtain:

$$\begin{aligned}
2^{-2} &\geq \sum_{l=1}^{n+1} \beta_l^{00} (\tfrac{1}{2})^l = 2^{-n-1}\beta_{n+1}^{00} + S(\,{}^0\mathcal{C}_n^0)\,, \\
2^{-3} &\geq \sum_{l=1}^{n+1} \beta_l^{ab} (\tfrac{1}{2})^l = 2^{-n-1}\beta_{n+1}^{ab} + S(\,{}^a\mathcal{C}_n^b) \quad \forall\, ab \in \{01, 10, 11\}\,.
\end{aligned}$$

¿From the above follows :

$$\begin{aligned}
\beta_{n+1}^{00} &\leq 2^{n-1} - 2^{n+1} S(\,{}^0\mathcal{C}_n^0)\,, \\
\beta_{n+1}^{01} &\leq 2^{n-2} - 2^{n+1} S(\,{}^0\mathcal{C}_n^1)\,, \\
\beta_{n+1}^{10} &\leq 2^{n-2} - 2^{n+1} S(\,{}^1\mathcal{C}_n^0)\,, \\
\beta_{n+1}^{11} &\leq 2^{n-2} - 2^{n+1} S(\,{}^1\mathcal{C}_n^1)\,.
\end{aligned} \tag{5.3}$$

By Proposition (66) (i) we obtain:

$$|a\mathcal{A}^{n-1}b - \Delta_B^{n+1}(\mathcal{C}_n)| \geq 2^{n-1} - |\Delta_P^n(\,{}^a\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^b)| \quad \forall\, a, b \in \{0,1\} \tag{5.4}$$

If $\beta_{n+1}^{ab} \leq 2^{n-1} - |\Delta_P^n(\,{}^a\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^b)|$ for $a, b \in \{0,1\}$ then from (5.4) follows that there are β_{n+1}^{ab} words in $a\mathcal{A}^{n-1}b$ which are not in the the bifix-shadow of

$\mathcal{C}_n$ by adding these codewords for every $a, b \in \{0,1\}$ to $\mathcal{C}_n$ we obtain a new fix-free code $\mathcal{C}_{n+1} \subseteq \bigcup_{l=1}^{n+1} \mathcal{A}^l$ with $|\mathcal{C}_{n+1} \cap \mathcal{A}^l| = \alpha_l$ and $|{}^a\mathcal{C}_{n+1}^b \cap \mathcal{A}^l| = \beta_l^{ab}$ for all $l \in \{1, \dots, n+1\}$, $a, b \in \{0,1\}$.

Therefore it is sufficient to show that :

$$\beta_{n+1}^{ab} \leq 2^{n-1} - |\Delta_P^n({}^a\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^b)| \quad \forall\, a, b \in \{0,1\} \text{ with } \beta_{n+1}^{ab} > 0 \tag{5.5}$$

By the definition of the $(\beta_l^{ab})_{l\in\mathbb{N}}$ we have to distinguish five cases:

Case 1:

$$S({}^0\mathcal{C}_n^0) = \sum_{l=1}^{n} \beta_l^{00} \left(\frac{1}{2}\right)^l < \frac{1}{4} \text{ and } S({}^a\mathcal{C}_n^b) = \sum_{l=1}^{n} \beta_l^{ab} \left(\frac{1}{2}\right)^l = 0 \quad \forall\, ac \in \{01, 10, 11\}$$

With Proposition 66 (ii) follows:

$$\begin{aligned}
|\Delta_P^n({}^0\mathcal{C}_n)| &= |\Delta_S^n(\mathcal{C}^0)| = 2^n \cdot S({}^0\mathcal{C}^0) \\
|\Delta_P^n({}^1\mathcal{C}_n)| &= |\Delta_S^n(\mathcal{C}_n^1)| = 0
\end{aligned}$$

We obtain that (5.5) holds for all $a, b \in \{0,1\}$, since by (5.3) follows:

$$\begin{aligned}
\beta_{n+1}^{00} &\leq 2^{n-1} - 2 \cdot 2^n \cdot S({}^0\mathcal{C}_n^0) = 2^{n-1} - |\Delta_P^n({}^0\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^0)|\,, \\
\beta_{n+1}^{01} &\leq 2^{n-2} = 2^{n-1} - 2^{n-2} < 2^{n-1} - 2^n \cdot S({}^0\mathcal{C}_n^0)\,, \\
&= 2^{n-1} - |\Delta_P^n({}^0\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^1)|\,, \\
\beta_{n+1}^{10} &\leq 2^{n-2} = 2^{n-1} - 2^{n-2} < 2^{n-1} - 2^n \cdot S({}^0\mathcal{C}_n^0)\,, \\
&= 2^{n-1} - |\Delta_P^n({}^1\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^0)|\,, \\
\beta_{n+1}^{11} &\leq 2^{n-2} < 2^{n-1} = 2^{n-1} - |\Delta_P^n({}^1\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^1)|\,.
\end{aligned}$$

Case 2:

$$\begin{aligned}
&S({}^0\mathcal{C}_n^0) = \sum_{l=1}^{n} \beta_l^{00} \left(\tfrac{1}{2}\right)^l = 2^{-2}\,, \quad S({}^0\mathcal{C}_n^1) = \sum_{l=1}^{n} \beta_l^{01} \left(\tfrac{1}{2}\right)^l < \tfrac{1}{8} \text{ and} \\
&S({}^a\mathcal{C}_n^b) = \sum_{l=1}^{n} \beta_l^{ab} \left(\tfrac{1}{2}\right)^l = 0 \quad \forall\, ab \in \{10, 11\}
\end{aligned}$$

In this case with Proposition 66 (ii) follows:

$$\begin{aligned}
&|\Delta_P^n({}^0\mathcal{C}_n)| = 2^n \cdot S({}^0\mathcal{C}_n^0) + 2^n \cdot S({}^0\mathcal{C}_n^1) = 2^{n-2} + 2^n \cdot S({}^0\mathcal{C}_n^1)\,, \\
&|\Delta_S^n(\mathcal{C}_n^0)| = 2^n \cdot S({}^0\mathcal{C}_n^0) = 2^{n-2}\,, \\
&|\Delta_P^n({}^1\mathcal{C}_n)| = 0 \text{ and } |\Delta_S^n(\mathcal{C}_n^1)| = 2^n \cdot S({}^0\mathcal{C}_n^1)
\end{aligned}$$

Once again in this case with (5.3) follows that (5.5) holds, because:

$$\begin{array}{rcl}\beta^{00}_{n+1} & = & 0 \\ \beta^{01}_{n+1} & \leq & 2^{n-2} - 2^{n+1} \cdot S(\,^0\mathcal{C}^1_n) = 2^{n-1} - 2^{n-2} - 2^n \cdot S(\,^0\mathcal{C}^1_n) - 2^n \cdot S(\,^0\mathcal{C}^1_n) \\ & = & 2^{n-1} - |\Delta^n_P(\,^0\mathcal{C}_n)| - |\Delta^n_S(\mathcal{C}^1_n)| \\ \beta^{10}_{n+1} & \leq & 2^{n-2} = 2^{n-1} - 2^{n-2} = 2^{n-1} - |\Delta^n_P(\,^1\mathcal{C}_n)| - |\Delta^n_S(\mathcal{C}^0_n)| \\ \beta^{11}_{n+1} & \leq & 2^{n-2} < 2^{n-1} - 2^{n-3} < 2^{n-1} - 2^n \cdot S(\,^0\mathcal{C}^1_n) \\ & = & 2^{n-1} - |\Delta^n_P(\,^1\mathcal{C}_n)| - |\Delta^n_S(\mathcal{C}^1_n)| \end{array}$$

Case 3:

$$\begin{array}{l} S(\,^0\mathcal{C}^0_n) = \sum\limits_{l=1}^{n} \beta^{00}_l \left(\frac{1}{2}\right)^l = 2^{-2}\,, \quad S(\,^0\mathcal{C}^1_n) = \sum\limits_{l=1}^{n} \beta^{01}_l \left(\frac{1}{2}\right)^l = 2^{-3}\,, \\ S(\,^1\mathcal{C}^0_n) = \sum\limits_{l=1}^{n} \beta^{10}_l \left(\frac{1}{2}\right)^l < 2^{-3} \text{ and } S(\,^1\mathcal{C}^1_n) = \sum\limits_{l=1}^{n} \beta^{11}_l \left(\frac{1}{2}\right)^l = 0 \end{array}$$

In this case with Proposition 66 (ii) follows:

$$\begin{array}{l} |\Delta^n_P(\,^0\mathcal{C}_n)| = 2^n \cdot S(\,^0\mathcal{C}^0_n) + 2^n \cdot S(\,^0\mathcal{C}^1_n) = 2^{n-2} + 2^{n-3}\,, \\ |\Delta^n_S(\mathcal{C}^0_n)| = 2^n \cdot S(\,^0\mathcal{C}^0_n) + 2^n \cdot S(\,^1\mathcal{C}^0_n) = 2^{n-2} + 2^n \cdot S(\,^1\mathcal{C}^0_n)\,, \\ |\Delta^n_P(\,^1\mathcal{C}_n)| = 2^n \cdot S(\,^1\mathcal{C}^0_n) \text{ and } |\Delta^n_S(\mathcal{C}^1_n)| = 2^n \cdot S(\,^0\mathcal{C}^1_n) = 2^{n-3} \end{array}$$

Also in this case (5.5) holds, because by (5.3) follows:

$$\begin{array}{rcl}\beta^{00}_{n+1} & = & \beta^{01}_{n+1} = 0 \\ \beta^{10}_{n+1} & \leq & 2^{n-2} - 2^{n+1} \cdot S(\,^1\mathcal{C}^0_n) = 2^{n-1} - 2^n \cdot S(\,^1\mathcal{C}^0_n) - 2^{n-2} - 2^n \cdot S(\,^1\mathcal{C}^0_n) \\ & = & 2^{n-1} - |\Delta^n_P(\,^1\mathcal{C}_n)| - |\Delta^n_S(\mathcal{C}^0_n)| \\ \beta^{11}_{n+1} & \leq & 2^{n-2} = 2^{n-1} - 2 \cdot 2^{n-3} < 2^{n-1} - 2^n \cdot S(\,^1\mathcal{C}^0_n) - 2^{n-3} \\ & = & 2^{n-1} - |\Delta^n_P(\,^1\mathcal{C}_n)| - |\Delta^n_S(\mathcal{C}^1_n)| \end{array}$$

Case 4:

$$\begin{array}{l} S(\,^0\mathcal{C}^0_n) = \sum\limits_{l=1}^{n} \beta^{00}_l \left(\frac{1}{2}\right)^l = 2^{-2}\,, \quad S(\,^0\mathcal{C}^1_n) = \sum\limits_{l=1}^{n} \beta^{01}_l \left(\frac{1}{2}\right)^l = 2^{-3}\,, \\ S(\,^1\mathcal{C}^0_n) = \sum\limits_{l=1}^{n} \beta^{10}_l \left(\frac{1}{2}\right)^l = 2^{-3} \text{ and } S(\,^1\mathcal{C}^1_n) = \sum\limits_{l=1}^{n} \beta^{11}_l \left(\frac{1}{2}\right)^l < 2^{-3} \end{array}$$

In this case with Proposition 66 (ii) follows:

$$\begin{array}{l} |\Delta^n_P(\,^0\mathcal{C}_n)| = |\Delta^n_S(\mathcal{C}^0_n)| = 2^n \cdot S(\,^0\mathcal{C}^0_n) + 2^n \cdot S(\,^0\mathcal{C}^1_n) = 2^{n-2} + 2^{n-3} \text{ and} \\ |\Delta^n_P(\,^1\mathcal{C}_n)| = |\Delta^n_S(\mathcal{C}^1_n)| = 2^n \cdot S(\,^1\mathcal{C}^0_n) + 2^n \cdot S(\,^1\mathcal{C}^1_n) = 2^{n-3} + 2^n \cdot S(\,^1\mathcal{C}^1_n) \end{array}$$

Also in this case (5.5) holds because with (5.3) follows:

$$\begin{aligned}
\beta_{n+1}^{00} &= \beta_{n+1}^{01} = \beta_{n+1}^{10} = 0 \\
\beta_{n+1}^{11} &\leq 2^{n-2} - 2^{n+1} \cdot S({}^1\mathcal{C}_n^1) = 2^{n-1} - 2 \cdot (2^{n-3} + 2^n \cdot S({}^1\mathcal{C}_n^1)) \\
&= 2^{n-1} - |\Delta_P^n({}^1\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^1)|
\end{aligned}$$

Case 5:

$$S({}^0\mathcal{C}_n^0) = \sum_{l=1}^{n} \beta_l^{00} \left(\tfrac{1}{2}\right)^l = 2^{-2} \quad \text{and} \quad S({}^a\mathcal{C}_n^b) = \sum_{l=1}^{n} \beta_l^{ab} \left(\tfrac{1}{2}\right)^l = 2^{-3} \quad \text{for} \quad ab \neq 00$$

Since (5.3), we obtain $\beta_{n+1}^{ab} = 0$ for all $ab \in \{00, 01, 10, 11\}$. **q.e.d**

One can try to generalize the above theorem for alphabets of arbitrary length. Let $\mathcal{A} = \{0, \ldots, q-1\}$ for some $q \geq 2$ and $(\alpha_l)_{l \in \mathbb{N}}$ be a sequence of nonnegative sequence with $\alpha_1 = 0\,,\ \alpha_2 \leq 2$ and $\sum\limits_{l=1}^{\infty} \alpha_l \cdot q^{-l} = \frac{3q^2-q}{2q^3}$. Let $\preceq$ be a linear ordering on $\mathcal{A}^2$ with leats element $00 \in \mathcal{A}^2$. It is easy to verify that there exists a unique set of sequences of nonnegative integers:

$$\{(\beta_l^{ab})_{l \in \mathbb{N}} \,|\, a, b \in \mathcal{A}\}$$

with the properties:

$$\begin{array}{ll}
\sum\limits_{l=1}^{\infty} \beta_l^{aa} \cdot q^{-l} = \frac{q-a}{q^3} & \forall\, a \in \mathcal{A} \\
\sum\limits_{l=1}^{\infty} \beta_l^{ab} \cdot q^{-l} = \frac{1}{q^3} & \forall\, a, b \in \mathcal{A}\,,\ a \neq b \\
\sum\limits_{a,b \in \mathcal{A}} \beta_l^{ab} = \alpha_l & \forall\ l \in \mathbb{N} \\
\beta_m^{ab} > 0 \Rightarrow \beta_l^{cd} = 0 & \forall\, cd \succ ab\,, l < m
\end{array} \tag{5.6}$$

For example let $\prec$ the lexicographic ordering on $\mathcal{A}$. This means for $ab, cd \in \mathcal{A}^2$:

$$ab \preceq cd \Leftrightarrow a \leq c \quad \text{or} \quad a = b, b \leq d\,.$$

If $\mathcal{A} = \{0, 1\}$ then the sequences $(\beta_l^{00})_{l \in \mathbb{N}}\,,\ (\beta_l^{01})_{l \in \mathbb{N}}\,,\ (\beta_l^{10})_{l \in \mathbb{N}}\,,\ (\beta_l^{11})_{l \in \mathbb{N}}$ in the proof of Theorem 21 have the above properties for the lexicograpic ordering.

Let $|\mathcal{A}| = 3$. If the first eight terms of $(\alpha_l)_{l\in\mathbb{N}}$ and the $(\beta_l^{ab})_{l\in\mathbb{N}}$ are given by:

$$\begin{array}{ll}
(\alpha_1,\ldots,\alpha_8) & := (0\,,0\,,2\,,4\,,22\,,6\,,394\,,276) \\
(\beta_1^{00},\ldots,\beta_8^{00}) & := (0\,,0\,,2\,,3\,,\ 0\ ,0\,,\ 0\ ,\ 0\ ) \\
(\beta_1^{01},\ldots,\beta_8^{01}) & := (0\,,0\,,0\,,1\,,\ 6\ ,0\,,\ 0\ ,\ 0\ ) \\
(\beta_1^{02},\ldots,\beta_8^{02}) & := (0\,,0\,,0\,,0\,,\ 9\ ,0\,,\ 0\ ,\ 0\ ) \\
(\beta_1^{10},\ldots,\beta_8^{10}) & := (0\,,0\,,0\,,0\,,\ 7\ ,6\,,\ 0\ ,\ 0\ ) \\
(\beta_1^{11},\ldots,\beta_8^{11}) & := (0\,,0\,,0\,,0\,,\ 0\ ,0\,,162\,,\ 0\ ) \\
(\beta_1^{12},\ldots,\beta_8^{12}) & := (0\,,0\,,0\,,0\,,\ 0\ ,0\,,\ 81\ ,\ 0\ ) \\
(\beta_1^{20},\ldots,\beta_8^{20}) & := (0\,,0\,,0\,,0\,,\ 0\ ,0\,,\ 81\ ,\ 0\ ) \\
(\beta_1^{21},\ldots,\beta_8^{21}) & := (0\,,0\,,0\,,0\,,\ 0\ ,0\,,\ 70\ ,\ 33\ ) \\
(\beta_1^{22},\ldots,\beta_8^{22}) & := (0\,,0\,,0\,,0\,,\ 0\ ,0\,,\ 0\ ,243)
\end{array}$$

and $\alpha_l = \beta_l^{00} = \ldots = \beta_l^{22} = 0$ for $l > 8$.

Then $\sum\limits_{l=1}^{\infty} \alpha_l \cdot 3^{-l} = \frac{12}{27} = \frac{3\cdot 3^2-3}{2\cdot 3^3}$ and $(\beta_l^{00})_{l\in\mathbb{N}},\ldots,(\beta_l^{22})_{l\in\mathbb{N}}$ are the unique sequences which have the properties in (5.6) for the lexicographic ordering.

Conjecture 4 *Let $q \geq 2$, $\mathcal{A} = \{0,\ldots,q-1\}$ and $(\alpha_l)_{l\in\mathbb{N}}$ be a sequence of nonnegative integers with $\alpha_1 = 0$, $\alpha_2 \leq 1$ and $\sum\limits_{l=1}^{\infty} \alpha_l \cdot q^{-l} = \frac{3q^2-q}{2q^3}$.*

(1) Then there exists a linear ordering $\preceq$ on $\mathcal{A}^2$ with least element 00 and a fix-free code $\mathcal{C} \subseteq \mathcal{A}^*$ with

$$|\,{}^a\mathcal{C}^b \cap \mathcal{A}_l| = \beta_l^{ab} \quad \forall\, l \in \mathbb{N},$$

where the $(\beta_l^{ab})_{l\in\mathbb{N}}$ are the unique sequences which fulfill (5.6) for $\preceq$.

(2) The first part of the conjecture holds for the lexicographic ordering of $\mathcal{A}^2$.

The conjecture above is a generalization of the idea of the proof of Theorem 21 Furthermore the proof of theorem 21 shows that both part of the conjecture holds for $q = 2$.

If part (1) of the conjecture holds for some $q \geq 2$ then from the second property of the $(\beta_l^{ab})_{l\in\mathbb{N}}$ follows that for every sequence $(\alpha_l)_{l\in\mathbb{N}}$ with $\alpha_1 = 0$, $\alpha_2 \leq 1$ and $\sum\limits_{l=1}^{\infty} \alpha_l \cdot q^{-l} \leq \frac{3q^2-q}{2q^3}$. there exists a fix-free code which fits to $(\alpha_l)_{l\in\mathbb{N}}$. On the other hand, this gives nothing new for $q \geq 3$, because $\frac{3q^2-q}{2q^3}$ is a decreasing sequence for $q \in \mathbb{N}$, as one can easy verify, and $\frac{3\cdot 3^2-3}{2\cdot 3^3} = \frac{24}{54} < \frac{1}{2}$. Indeed Theorem 19 says

already that for every sequence $(\alpha_l)_{l\in\mathbb{N}}$ with Kraftsum smaller than or equal to $\frac{1}{2}$ there exist a fix-free Code $\mathcal{C} \subseteq \mathcal{A}^*$ which fits to $(\alpha_l)_{l\in\mathbb{N}}$. The only new, is the special form of the fix-free code. Therefore we omit a full proof of the conjecture and finish this section, by showing that both part of the conjecture holds for $q = 3$.

Proof of the Conjecture for $q = 3$ **:** Let $q = 3$, $\mathcal{A} = \{0, 1, 2\}$ and $\preceq$ the lexicographic ordering on $\mathcal{A}^2$. Let $(\alpha_l)_{l\in\mathbb{N}}$, $(\beta_l^{00})_{l\in\mathbb{N}}, \dots, (\beta_l^{22})_{l\in\mathbb{N}}$ as in the conjecture. The proof of the conjecture will be similar to the proof of Theorem 21. This means, we will construct by induction, fix-free sets $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3 \subseteq \dots$ with the property:

$$\mathcal{C}_n \subseteq \bigcup_{l=1}^{n} \mathcal{A}^l \text{ and } |{}^a\mathcal{C}_n^b \cap \mathcal{A}^l| = \beta_l^{ab} \quad \forall\, l \in \{1, \dots, n\},\ a, b \in \mathcal{A} \tag{5.7}$$

Then $\mathcal{C} := \bigcup\limits_{l=1}^{\infty} \mathcal{C}_n$ is a fix-free code for which the conditions of the conjecture holds.

If $\mathcal{C}_1 := \emptyset$ then $\mathcal{C}_1$ is a fix-free set for which (5.7) hold.
Let $\mathcal{C}_n$ be a fix-free set for which (5.7) holds. Then we obtain with (5.6):

$$\begin{aligned}
3^{-2} \quad &= \quad 3 \cdot 3^{-3} \geq \sum_{l=1}^{n+1} \beta_l^{00} \cdot q^{-l} = 3^{-n-1}\beta_{n+1}^{00} + S({}^0\mathcal{C}^0)\,, \\
2 \cdot 3^{-3} \quad &\geq \quad \sum_{l=1}^{n+1} \beta_l^{11} \cdot q^{-l} = 3^{-n-1}\beta_{n+1}^{11} + S({}^1\mathcal{C}^1) \\
3^{-3} \quad &\geq \quad \sum_{l=1}^{n+1} \beta_l^{22} \cdot q^{-l} = 3^{-n-1}\beta_{n+1}^{22} + S({}^2\mathcal{C}^2) \\
3^{-3} \quad &\geq \quad \sum_{l=1}^{n+1} \beta_l^{ab} \cdot q^{-l} = 3^{-n-1}\beta_{n+1}^{ab} + S({}^a\mathcal{C}^b) \quad \forall\, a, b \in \{0, 1, 2\}\,,\ a \neq b\,.
\end{aligned}$$

With this follows:

$$\begin{array}{lcl}
\beta_{n+1}^{00} \leq 3^{n-1} - 3^{n+1} \cdot S({}^0\mathcal{C}^0) & ; & \beta_{n+1}^{01} \leq 3^{n-2} - 3^{n+1} \cdot S({}^0\mathcal{C}^1)\,; \\
\beta_{n+1}^{02} \leq 3^{n-2} - 3^{n+1} \cdot S({}^0\mathcal{C}^2) & ; & \beta_{n+1}^{10} \leq 3^{n-2} - 3^{n+1} \cdot S({}^1\mathcal{C}^0)\,; \\
\beta_{n+1}^{11} \leq 2 \cdot 3^{n-2} - 3^{n+1} \cdot S({}^1\mathcal{C}^1) & ; & \beta_{n+1}^{12} \leq 3^{n-2} - 3^{n+1} \cdot S({}^1\mathcal{C}^2)\,; \\
\beta_{n+1}^{20} \leq 3^{n-2} - 3^{n+1} \cdot S({}^2\mathcal{C}^0) & ; & \beta_{n+1}^{21} \leq 3^{n-2} - 3^{n+1} \cdot S({}^2\mathcal{C}^1) \text{ and} \\
\beta_{n+1}^{22} \leq 3^{n-2} - 3^{n+1} \cdot S({}^2\mathcal{C}^2) & &
\end{array} \tag{5.8}$$

By Proposition 66 (i) we have:

$$|a\mathcal{A}^{n-2}b - \Delta_B^{n+1}(\mathcal{C}_n)| \geq \max\left\{0, 3^{n-1} - |\Delta_P^n({}^a\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^b)|\right\} \quad \forall\, a, b \in \{0, 1, 2\} \tag{5.9}$$

Therefore it follows, that for the existence of a fix-free set $\mathcal{C}_{n+1} \supseteq \mathcal{C}_n$ with property (5.7), it is sufficient to show that:

$$\beta_{n+1}^{ab} \leq \max\left\{0, 3^{n-1} - |\Delta_P^n(\,^a\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^b)|\right\} \quad \forall\, a, b \in \{0,1,2\} \text{ with } \beta_{n+1}^{ab} > 0 \tag{5.10}$$

If 5.10 holds, then by (5.9) and 5.10) follows, that for all $a, b \in \mathcal{A}$ there exist β_{n+1}^{ab} codewords in $a\mathcal{A}^{n-1}b$ which are not in the bifix-shadow of $\mathcal{C}_n$. By adding these codewords to $\mathcal{C}_n$ we obtain a fix-free code $\mathcal{C}_{n+1} \supseteq \mathcal{C}_n$ for which (5.7) holds. To show (5.10) we have to distinguish ten cases:

Case 1:

$$S(\,^0\mathcal{C}_n^0) < 3^{-2} \text{ and } S(\,^a\mathcal{C}_n^b) = 0 \quad \forall\, a, b \in \mathcal{A}\,,\ ab \neq 00$$

By Proposition 66 (ii) we obtain:

$$\begin{array}{ll} |\Delta_P^n(\,^0\mathcal{C}_n)| = |\Delta_S^n(\mathcal{C}_n^0)| = 3^n S(\,^0\mathcal{C}_n^0) < 3^{n-2} & \\ |\Delta_P^n(\,^a\mathcal{C}_n)| = |\Delta_S^n(\mathcal{C}_n^b)| = 0 & \forall\, a, b \in \mathcal{A}\,,\ a, b \neq 0 \end{array}$$

¿From (5.8) follows that (5.10) holds, because:

$$\begin{array}{rrcl} & \beta_{n+1}^{00} & \leq & 3^{n-1} - 3^{n+1} \cdot S(\,^0\mathcal{C}_n^0) < 3^{n-1} - 2 \cdot 3^n \cdot S(\,^0\mathcal{C}_n^0) \\ & & = & 3^{n-1} - |\Delta_P^n(\,^0\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^0)| \\ \forall\, b \neq 0: & \beta_{n+1}^{0b} & \leq & 3^{n-2} < 3^{n-1} - 3^{n-2} \leq 3^{n-1} - 3^n \cdot S(\,^0\mathcal{C}_n^0) \\ & & = & 3^{n-1} - |\Delta_P^n(\,^0\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^b)| \\ & \beta_{n+1}^{11} & \leq & 2 \cdot 3^{n-2} < 3^{n-1} = 3^{n-1} - |\Delta_P^n(\,^1\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^1)| \\ \forall\, b \neq 1: & \beta_{n+1}^{1b} & \leq & 3^{n-2} < 3^{n-1} - 3^{n-2} \leq 3^{n-1} - |\Delta_P^n(\,^1\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^b)| \\ \forall\, b \in \mathcal{A}: & \beta_{n+1}^{2b} & \leq & 3^{n-2} < 3^{n-1} - 3^{n-2} \leq 3^{n-1} - |\Delta_P^n(\,^2\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^b)| \end{array}$$

Case 2:

$$\begin{array}{l} S(\,^0\mathcal{C}_n^0) = 3^{-2}\,,\ S(\,^0\mathcal{C}_n^1) < 3^{-3} \\ S(\,^a\mathcal{C}_n^b) = 0 \quad \forall\, ab \in \mathcal{A}^2 - \{00, 01\} \end{array}$$

In this case we obtain with proposition 66 (ii):

$$\begin{array}{lcl} |\Delta_P^n(\,^0\mathcal{C}_n)| & = & 3^{n-2} + 3^n \cdot S(\,^0\mathcal{C}_n^1) < 2 \cdot 3^{n-2}\,,\ |\Delta_P^n(\,^a\mathcal{C}_n)| = 0 \quad \forall\, a \geq 1\,, \\ |\Delta_S^n(\mathcal{C}_n^0)| & = & 3^{n-2}\,,\ |\Delta_S^n(C_n^1)| = 3^n \cdot S(\,^0\mathcal{C}_n^1) < 3^{n-3} \text{ and } |\Delta_S^n(\mathcal{C}_n^2)| = 0 \end{array}$$

By (5.8) follows (5.10):

$$\begin{array}{rrcl}
& \beta_{n+1}^{00} & = & 0 \\
& \beta_{n+1}^{01} & \leq & 3^{n-2} - 3^{n+1} \cdot S({}^{0}\mathcal{C}_n^1) \leq 3^{n-1} - 3^{n-2} - 2 \cdot 3^n \cdot S({}^{0}\mathcal{C}_n^1) \\
& & = & 3^{n-1} - |\Delta_P^n({}^{0}\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^1)| \\
& \beta_{n+1}^{02} & \leq & 3^{n-2} = 3^{n-1} - 2 \cdot 3^{n-2} \leq 3^{n-1} - 3^{n-2} - 3^n \cdot S({}^{0}\mathcal{C}_n^2) \\
& & = & 3^{n-1} - |\Delta_P^n({}^{0}\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^2)| \\
& \beta_{n+1}^{11} & \leq & 2 \cdot 3^{n-2} < 3^{n-1} - 3^{n-3} \leq 3^{n-1} - |\Delta_P^n({}^{1}\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^1)| \\
& \beta_{n+1}^{12} & \leq & 3^{n-2} < 3^{n-1} = 3^{n-1} - |\Delta_P^n({}^{1}\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^2)| \\
\forall\, b \in \mathcal{A} & \beta_{n+1}^{2b} & \leq & 3^{n-2} < 3^{n-1} - 3^{n-2} \leq 3^{n-1} - |\Delta_P^n({}^{2}\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^b)|
\end{array}$$

Case 3:

$$\begin{array}{l}
S({}^{0}\mathcal{C}_n^0) = 3^{-2}\,,\ S({}^{0}\mathcal{C}_n^1) = 3^{-3}\,,\ S({}^{0}\mathcal{C}_n^2) < 3^{-3}\,, \\
S({}^{a}\mathcal{C}_n^b) = 0 \quad \forall\, a, b \in \mathcal{A} - \{00, 01, 02\}
\end{array}$$

In this case we obtain by proposition 66 (ii):

$$\begin{array}{lcl}
|\Delta_P^n({}^{0}\mathcal{C}_n)| & = & 3^{n-2} + 3^{n-3} + 3^n \cdot S({}^{0}\mathcal{C}_n^2) < 5 \cdot 3^{n-3}\,,\ |\Delta_P^n({}^{a}\mathcal{C}_n)| = 0 \quad \forall\, a \geq 1\,, \\
|\Delta_S^n(\mathcal{C}_n^0)| & = & 3^{n-2}\,,\ |\Delta_S^n(\mathcal{C}_n^1)| = 3^{n-3} \ \text{ and } \ |\Delta_S^n(\mathcal{C}_n^2)| = 3^n \cdot S({}^{0}\mathcal{C}_n^2) < 3^{n-3}
\end{array}$$

Once again with (5.8) follows (5.10):

$$\begin{array}{rrcl}
& \beta_{n+1}^{00} & = & \beta_{n+1}^{01} = 0 \\
& \beta_{n+1}^{02} & \leq & 3^{n-2} - 3^{n+1} \cdot S({}^{0}\mathcal{C}_n^2) \\
& & < & 3^{n-1} - 3^{n-2} - 3^{n-3} - 3^n \cdot S({}^{0}\mathcal{C}_n^2) - 3^n \cdot S({}^{0}\mathcal{C}_n^2) \\
& & = & 3^{n-1} - |\Delta_P^n({}^{0}\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^1)| \\
& \beta_{n+1}^{11} & \leq & 2 \cdot 3^{n-2} < 8 \cdot 3^{n-3} = 3^{n-1} - 3^{n-3} \\
& & < & 3^{n-1} - |\Delta_P^n({}^{1}\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^1)| \\
& \beta_{n+1}^{12} & \leq & 3^{n-2} < 3^{n-1} - 3^{n-3} \\
& & < & 3^{n-1} - |\Delta_P^n({}^{1}\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^2)| \\
\forall\, b \in \mathcal{A}\ : & \beta_{n+1}^{2b} & \leq & 3^{n-2} < 3^{n-1} - 3^{n-2} \\
& & < & 3^{n-1} - |\Delta_P^n({}^{2}\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^b)|
\end{array}$$

Case 4:

$$\begin{array}{l}
S({}^{0}\mathcal{C}_n^0) = 3^{-2}\,,\ S({}^{0}\mathcal{C}_n^1) = S({}^{0}\mathcal{C}_n^2) = 3^{-3}\,, \\
S({}^{1}\mathcal{C}_n^0) < 3^{-3}\,,\ S({}^{a}\mathcal{C}_n^b) = 0 \quad \forall\, ab \in \mathcal{A} - \{00, 01, 02, 10\}
\end{array}$$

In this case we obtain with proposition 66 (ii):

$$\begin{array}{rcl}
|\Delta_P^n({}^0\mathcal{C}_n)| &=& 3^{n-2}+2\cdot 3^{n-3}=4\cdot 3^{n-3}\,,\ |\Delta_P^n({}^1\mathcal{C}_n)|=3^nS({}^1\mathcal{C}_n^0)<3^{n-3}\,,\\
|\Delta_P^n({}^2\mathcal{C}_n)| &=& 0\,,\ |\Delta_S^n(\mathcal{C}_n^0)|=3^{n-2}+3^n\cdot S({}^1\mathcal{C}_n^0)<4\cdot 3^{n-3}\,,\ |\Delta_S^n(\mathcal{C}_n^1)|=3^{n-3}\\
|\Delta_S^n(\mathcal{C}_n^2)| &=& 3^{n-3}
\end{array}$$

Once again with (5.8) follows (5.10):

$$\begin{array}{rrcl}
& \beta_{n+1}^{00} &=& \beta_{n+1}^{01}=\beta_{n+1}^{02}=0\\
& \beta_{n+1}^{10} &\le& 3^{n-2}-3^{n+1}\cdot S({}^1\mathcal{C}_n^0)\\
& &<& 3^{n-1}-3^n\cdot S({}^1\mathcal{C}_n^0)-3^{n-2}-3^n\cdot S({}^1\mathcal{C}_n^0)\\
& &=& 3^{n-1}-|\Delta_P^n({}^1\mathcal{C}_n)|-|\Delta_S^n(\mathcal{C}_n^0)|\\
& \beta_{n+1}^{11} &\le& 2\cdot 3^{n-2}<7\cdot 3^{n-3}=3^{n-1}-2\cdot 3^{n-3}\\
& &<& 3^{n-1}-|\Delta_P^n({}^1\mathcal{C}_n)|-|\Delta_S^n(\mathcal{C}_n^1)|\\
& \beta_{n+1}^{12} &\le& 3^{n-2}<7\cdot 3^{n-3}=3^{n-1}-2\cdot 3^{n-3}\\
& &<& 3^{n-1}-|\Delta_P^n({}^1\mathcal{C}_n)|-|\Delta_S^n(\mathcal{C}_n^2)|\\
& \beta_{n+1}^{20} &\le& 3^{n-2}<5\cdot 3^{n-3}=3^{n-1}-4\cdot 3^{n-3}\\
& &<& 3^{n-1}-|\Delta_P^n({}^2\mathcal{C}_n)|-|\Delta_S^n(\mathcal{C}_n^0)|\\
\forall\, b\ge 1 & \beta_{n+1}^{2b} &\le& 3^{n-2}<8\cdot 3^{n-3}=3^{n-1}-3^{n-3}=3^{n-1}-|\Delta_P^n({}^2\mathcal{C}_n)|-|\Delta_S^n(\mathcal{C}_n^b)|
\end{array}$$

Case 5:

$$\begin{array}{l}
S({}^0\mathcal{C}_n^0)=3^{-2}\,,\ S({}^0\mathcal{C}_n^1)=S({}^0\mathcal{C}_n^2)=S({}^1\mathcal{C}_n^0)=3^{-3}\,,\\
S({}^1\mathcal{C}_n^1)<2\cdot 3^{-3}\,,\ S({}^a\mathcal{C}_n^b)=0\quad\forall\, ab\in\{12,20,21,22\}
\end{array}$$

In this case we obtain with proposition 66 (ii):

$$\begin{array}{rclcrcl}
|\Delta_P^n({}^0\mathcal{C}_n)| &=& 3^{n-2}+2\cdot 3^{n-3}=5\cdot 3^{n-3} &,& |\Delta_S^n(\mathcal{C}_n^0)| &=& 3^{n-2}+3^{n-3}=4\cdot 3^{n-3}\\
|\Delta_P^n({}^1\mathcal{C}_n)| &=& 3^{n-3}+3^n\cdot S({}^1\mathcal{C}_n^1)<3^{n-2} &,& |\Delta_S^n(\mathcal{C}_n^1)| &=& 3^{n-3}+3^n\cdot S({}^1\mathcal{C}_n^1)<3^{n-2}\,,\\
|\Delta_P^n({}^2\mathcal{C}_n)| &=& 0 &,& |\Delta_S^n(\mathcal{C}_n^2)| &=& 3^{n-3}
\end{array}$$

Also in this case follows (5.10) with (5.8):

$$\begin{array}{rrcl}
& \beta_{n+1}^{00} &=& \beta_{n+1}^{01}=\beta_{n+1}^{02}=\beta_{n+1}^{10}=0\\
& \beta_{n+1}^{11} &\le& 2\cdot 3^{n-2}-3^{n+1}\cdot S({}^1\mathcal{C}_n^1)<3^{n-1}-2\cdot 3^{n-3}-2\cdot 3^nS({}^1\mathcal{C}_n^1)\\
& &=& 3^{n-1}-|\Delta_P^n({}^1\mathcal{C}_n)|-|\Delta_S^n(\mathcal{C}_n^1)|\\
& \beta_{n+1}^{12} &\le& 3^{n-2}<^5\cdot 3^{n-3}=3^{n-1}-4\cdot 3^{n-3}\\
& &<& 3^{n-1}-|\Delta_P^n({}^1\mathcal{C}_n)|-|\Delta_S^n(\mathcal{C}_n^2)|\\
\forall\, b\in\mathcal{A} & \beta_{n+1}^{2b} &\le& 3^{n-2}<5\cdot 3^{n-3}=3^{n-1}-4\cdot 3^{n-3}\\
& &\le& 3^{n-1}-|\Delta_P^n({}^2\mathcal{C}_n)|-|\Delta_S^n(\mathcal{C}_n^b)|
\end{array}$$

Case 6:

$$S({}^0\mathcal{C}_n^0) = 3^{-2}\,,\ S({}^1\mathcal{C}_n^1) = 2\cdot 3^{-3}\,,\ S({}^0\mathcal{C}_n^1) = S({}^0\mathcal{C}_n^2) = S({}^1\mathcal{C}_n^0) = 3^{-3}\,,$$
$$S({}^1\mathcal{C}_n^2) < 3^{-3}\,,\ S({}^2\mathcal{C}_n^b) = 0 \quad \forall\, b \in \mathcal{A}$$

In this case we obtain with proposition 66 (ii):

$$\begin{array}{lcl}
|\Delta_P^n({}^0\mathcal{C}_n)| & = & 3^{n-2} + 2\cdot 3^{n-3} = 5\cdot 3^{n-3} \\
|\Delta_P^n({}^1\mathcal{C}_n)| & = & 3\cdot 3^{n-3} + 3^n\cdot S({}^1\mathcal{C}_n^2) < 4\cdot 3^{n-3} \\
|\Delta_P^n({}^2\mathcal{C}_n)| & = & 0 \\
|\Delta_S^n(\mathcal{C}_n^0)| & = & 3^{n-2} + 3^{n-3} = 4\cdot 3^{n-3} \\
|\Delta_S^n(\mathcal{C}_n^1)| & = & 3\cdot 3^{n-3} = 3^{n-2} \\
|\Delta_S^n(\mathcal{C}_n^2)| & = & 3^{n-3} + 3^n\cdot S({}^1\mathcal{C}_n^2) < 2\cdot 3^{n-3}
\end{array}$$

Now (5.10) follows with (5.8):

$$\begin{array}{rlcl}
 & \beta_{n+1}^{00} & = & \beta_{n+1}^{01} = \beta_{n+1}^{02} = \beta_{n+1}^{10} = \beta_{n+1}^{11} = 0 \\
 & \beta_{n+1}^{12} & \le & 3^{n-2} - 3^{n+1}\cdot S({}^1\mathcal{C}_n^2) < 5\cdot 3^{n-3} - 2\cdot 3^n S({}^1\mathcal{C}_n^2) \\
 & & = & 3^{n-1} - |\Delta_P^n({}^1\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^2)| \\
\forall\, b \in \mathcal{A} & \beta_{n+1}^{2b} & \le & 3^{n-2} < 5\cdot 3^{n-3} = 3^{n-1} - 4\cdot 3^{n-3} \\
 & & \le & 3^{n-1} - |\Delta_P^n({}^2\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^b)|
\end{array}$$

Case 7:

$$S({}^0\mathcal{C}_n^0) = 3^{-2}\,,\ S({}^1\mathcal{C}_n^1) = 2\cdot 3^{-3}\,,\ S({}^0\mathcal{C}_n^1) = S({}^0\mathcal{C}_n^2) = S({}^1\mathcal{C}_n^0) = S({}^1\mathcal{C}_n^2) = 3^{-3}\,,$$
$$S({}^2\mathcal{C}_n^0) < 3^{-3}\,,\ S({}^2\mathcal{C}_n^b) = 0 \quad \forall\, b \in \{1,2\}$$

In this case we obtain with proposition 66 (ii):

$$\begin{array}{lcl}
|\Delta_P^n({}^0\mathcal{C}_n)| & = & 3^{n-2} + 2\cdot 3^{n-3} = 5\cdot 3^{n-3} \\
|\Delta_P^n({}^1\mathcal{C}_n)| & = & 4\cdot 3^{n-3} \\
|\Delta_P^n({}^2\mathcal{C}_n)| & = & 3^n\cdot S({}^2\mathcal{C}_n^0) < 3^{n-3} \\
|\Delta_S^n(\mathcal{C}_n^0)| & = & 4\cdot 3^{n-3} + 3^n\cdot S({}^2\mathcal{C}_n^0) < 5\cdot 3^{n-3} \\
|\Delta_S^n(\mathcal{C}_n^1)| & = & 3\cdot 3^{n-3} = 3^{n-2} \\
|\Delta_S^n(\mathcal{C}_n^2)| & = & 2\cdot 3^{n-3}
\end{array}$$

Now (5.10) follows with (5.8):

$$\begin{array}{lcl}
\beta_{n+1}^{00} & = & \beta_{n+1}^{01} = \beta_{n+1}^{02} = \beta_{n+1}^{10} = \beta_{n+1}^{11}\beta_{n+1}^{12} = 0 \\
\beta_{n+1}^{20} & \le & 3^{n-2} - 3^{n+1}\cdot S({}^2\mathcal{C}_n^0) < 5\cdot 3^{n-3} - 2\cdot 3^n S({}^2\mathcal{C}_n^0) \\
 & = & 3^{n-1} - |\Delta_P^n({}^2\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^0)| \\
\beta_{n+1}^{21} & \le & 3^{n-2} < 5\cdot 3^{n-3} = 3^{n-1} - 4\cdot 3^{n-3} \\
 & < & 3^{n-1} - |\Delta_P^n({}^2\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^1)| \\
\beta_{n+1}^{22} & \le & 3^{n-2} < 5\cdot 3^{n-3} = 3^{n-1} - 4\cdot 3^{n-3} \\
 & < & 3^{n-1} - |\Delta_P^n({}^2\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^2)|
\end{array}$$

Case 8:

$$S({}^0\mathcal{C}_n^0) = 3^{-2}\,,\ S({}^1\mathcal{C}_n^1) = 2\cdot 3^{-3}\,,\ S({}^2\mathcal{C}_n^2) = 0\,,\ S({}^2\mathcal{C}_n^1) < 3^{-3}$$
$$S({}^0\mathcal{C}_n^1) = S({}^0\mathcal{C}_n^2) = S({}^1\mathcal{C}_n^0) = S({}^1\mathcal{C}_n^2) = S({}^2\mathcal{C}_n^0) = 3^{-3}$$

In this case we obtain with proposition 66 (ii):

$$\begin{array}{lcl}
|\Delta_P^n({}^0\mathcal{C}_n)| &=& 3^{n-2} + 2\cdot 3^{n-3} = 5\cdot 3^{n-3}\\
|\Delta_P^n({}^1\mathcal{C}_n)| &=& 4\cdot 3^{n-3}\\
|\Delta_P^n({}^2\mathcal{C}_n)| &=& 3^{n-3} + 3^n\cdot S({}^2\mathcal{C}_n^1) < 2\cdot 3^{n-3}\\
|\Delta_S^n(\mathcal{C}_n^0)| &=& 5\cdot 3^{n-3}\\
|\Delta_S^n(\mathcal{C}_n^1)| &=& 3\cdot 3^{n-3} + 3^n\cdot S({}^2\mathcal{C}_n^1) < 4\cdot 3^{n-3}\\
|\Delta_S^n(\mathcal{C}_n^2)| &=& 2\cdot 3^{n-3}
\end{array}$$

Now (5.10) follows with (5.8):

$$\begin{array}{lcl}
\beta_{n+1}^{00} &=& \beta_{n+1}^{01} = \beta_{n+1}^{02} = \beta_{n+1}^{10} = \beta_{n+1}^{11}\beta_{n+1}^{12} = \beta_{n+1}^{20} = 0\\
\beta_{n+1}^{21} &\le& 3^{n-2} - 3^{n+1}\cdot S({}^2\mathcal{C}_n^1) < 5\cdot 3^{n-3} - 2\cdot 3^n\cdot S({}^2\mathcal{C}_n^1)\\
&=& 3^{n-1} - |\Delta_P^n({}^2\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^1)|\\
\beta_{n+1}^{22} &\le& 3^{n-2} < 5\cdot 3^{n-3} < 3^{n-1} - |\Delta_P^n({}^2\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^2)|
\end{array}$$

Case 9:

$$S({}^0\mathcal{C}_n^0) = 3^{-2}\,,\ S({}^1\mathcal{C}_n^1) = 2\cdot 3^{-3}\,,\ S({}^2\mathcal{C}_n^2) < 3^{-3}\,,$$
$$S({}^0\mathcal{C}_n^1) = S({}^0\mathcal{C}_n^2) = S({}^1\mathcal{C}_n^0) = S({}^1\mathcal{C}_n^2) = S({}^2\mathcal{C}_n^0) = S({}^2\mathcal{C}_n^1) = 3^{-3}$$

In this case we obtain with proposition 66 (ii):

$$\begin{array}{lcl}
|\Delta_P^n({}^0\mathcal{C}_n)| &=& 3^{n-2} + 2\cdot 3^{n-3} = 5\cdot 3^{n-3}\\
|\Delta_P^n({}^1\mathcal{C}_n)| &=& 4\cdot 3^{n-3}\\
|\Delta_P^n({}^2\mathcal{C}_n)| &=& 2\cdot 3^{n-3} + 3^n\cdot S({}^2\mathcal{C}_n^2) < 3^{n-2}\\
|\Delta_S^n(\mathcal{C}_n^0)| &=& 5\cdot 3^{n-3}\\
|\Delta_S^n(\mathcal{C}_n^1)| &=& 4\cdot 3^{n-3}\\
|\Delta_S^n(\mathcal{C}_n^2)| &=& 2\cdot 3^{n-3} + 3^n\cdot S({}^2\mathcal{C}_n^2) < 3^{n-2}
\end{array}$$

Also in this case (5.10) follows with (5.8):

$$\begin{array}{lcl}
\beta_{n+1}^{00} &=& \beta_{n+1}^{01} = \beta_{n+1}^{02} = \beta_{n+1}^{10} = \beta_{n+1}^{11}\beta_{n+1}^{12} = \beta_{n+1}^{20} = \beta_{n+1}^{21} = 0\\
\beta_{n+1}^{22} &\le& 3^{n-2} - 3^{n+1}\cdot S({}^2\mathcal{C}_n^2) < 5\cdot 3^{n-3} - 2\cdot 3^n\cdot S({}^2\mathcal{C}_n^2)\\
&=& 3^{n-1} - |\Delta_P^n({}^2\mathcal{C}_n)| - |\Delta_S^n(\mathcal{C}_n^1)|
\end{array}$$

Case 10:

$$S({}^0\mathcal{C}_n^0) = 3^{-2}\,,\ S({}^1\mathcal{C}_n^1) = 2\cdot 3^{-3}\,,$$
$$S({}^0\mathcal{C}_n^1) = S({}^0\mathcal{C}_n^2) = S({}^1\mathcal{C}_n^0) = S({}^1\mathcal{C}_n^2) = S({}^2\mathcal{C}_n^0) = S({}^2\mathcal{C}_n^1) = S({}^2\mathcal{C}_n^2) = 3^{-3}$$

In this case we obtain $\beta_{n+1}^{ab} = 0$ for all $a, b \in \mathcal{A}$. **q.e.d**

Appendix A

Overview of known results about the $\frac{3}{4}$-conjecture

In the appendix we give a collection of all known results about the $\frac{3}{4}$-conjecture up to now. Throughout the appendix we denote with $\mathcal{A}$ a finite alphabet with $|\mathcal{A}| \geq 2$ and $(\alpha_l)_{l\in\mathbb{N}}$ should be a finite sequence of nonnegative integers. We write a set $\mathcal{C} \subseteq \mathcal{A}^+$ fits to $(\alpha_l)_{l\in\mathbb{N}}$, if $|\mathcal{C} \cap \mathcal{A}^l| = \alpha_l$ for all $l \in \mathbb{N}$.

Theorem 22 (Kraft and McMillan [1]) *If $\mathcal{C} \subseteq \mathcal{A}^+$ is a code which fits to $(\alpha_l)_{l\in\mathbb{N}}$, then $\sum\limits_{l=1}^{\infty} \alpha_l q^{-l} \leq 1$.*

For prefix-free codes also the other direction of the theorem above holds.

Theorem 23 (Kraft and McMillan [1]) *$\sum\limits_{l=1}^{\infty} \alpha_l q^{-l} \leq 1$ if and only if there exists a prefix-free code which fits to $(\alpha_l)_{l\in\mathbb{N}}$.*

Krafts theorem holds also for suffix-free codes.

The $\frac{3}{4}$-conjecture is a possible generalization of the second implication in Krafts theorem for fix-free codes. The first implication is Theorem 22, which holds for all codes.

Conjecture 5 (Ahlswede, Balkenhol and Khachatrian) *If $\sum\limits_{l=1}^{\infty} \alpha_l q^{-l} \leq \frac{3}{4}$, then there exists a fix-free code which fits to $(\alpha_l)_{l\in\mathbb{N}}$.*

We distinguish theorems for the binary case $|\mathcal{A}| = 2$ and the general case $|\mathcal{A}| = q$ for some $q \geq 2$. The next theorem shows, that for sequences with Kraftsum bigger than $\frac{3}{4}$ the conjecture can not holds, but that the conjecture holds for sequences with Kraftsum smaller than or equal to $\frac{1}{2}$.

Theorem 24
(Binary case : Ahlswede, Balkenhol and Khachatrian [5]
General case : Harada and Kobayashi [6])

(i) If $\sum_{l=1}^{\infty} \alpha_l q^{-l} \le \frac{1}{2}$*, then there exists a fix-free code which fits to* $(\alpha_n)_{n\in\mathbb{N}}$*.*

(ii) For every $\gamma > \frac{3}{4}$ *there exists a sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *with*

$$\frac{3}{4} < \sum_{l=1}^{\infty} \alpha_l q^{-l} \le \gamma \,,$$

such that there doesn't exist a fix-free code which fits to $(\alpha_n)_{n\in\mathbb{N}}$*.*

The next theorem shows, for which sequences the $\frac{3}{4}$-conjecture is already proven.

Theorem 25 *Let* $\sum_{l=1}^{\infty} \alpha_l q^{-l} \le \frac{3}{4}$*.*

(i) **(Binary case : Ahlswede, Balkenhol and Khachatrian [5]**
General case : Harada and Kobayashi [6])
If $2k \le \inf\{l \,|\, \alpha_l \neq 0\,, l > k\}$ *for all* $k \in \mathbb{N}$ *with* $\alpha_k \neq 0$*, then there exists a fix-free code which fits to* $(\alpha_n)_{n\in\mathbb{N}}$*.*

(ii) **(General case : Harada and Kobayashi [6])**
If there exists $n, m \in \mathbb{N}$ *such that* $\alpha_l = 0$ *for all* $l \notin \{n, m\}$*, then there exists a fix-free code which fits to* $(\alpha_n)_{n\in\mathbb{N}}$*.*

(iii) **(Binary case : Kukorelly and Zeger [10])**
General case : This survey, Chapter 2
Let $l_{min} := \min\{l|\alpha_l > 0\}$ *and* $l_{max} := sup\{l \in \mathbb{N}|\alpha_l > 0\} \le \infty$*. If* $l_{min} \ge 2$*,* $l_{max} < \infty$ *and* $\alpha_l \le q^{l_{min}-2}\lfloor\frac{q}{2}\rfloor^2\lceil\frac{q}{2}\rceil^{l-l_{min}}$ *for all* $l \neq l_{max}$*, then there exists a fix-free code which fits to* $(\alpha_n)_{n\in\mathbb{N}}$*.*

(iv) **(Binary case : Yekhanin [8] without a full proof.**
A full proof is in this survey Chapter 4.)
Let $|\mathcal{A}| = q = 2$ *and* $n := \min\{l \,|\, \alpha_l \neq 0\}$*. If* $\frac{\alpha_n}{2^n} + \frac{\alpha_{n+1}}{2^{n+1}} \ge \frac{1}{2}$*, then there exists a fix-free code which fits to* $(\alpha_n)_{n\in\mathbb{N}}$*.*

(v) **(Binary case : See (iv).**
General case : This survey, Chapter 4.)
Let $\frac{\alpha_n}{q^n} + \frac{\alpha_{n+1}}{q^{n+1}} \ge \lceil\frac{q}{2}\rceil\frac{1}{q}$*.*
If $\alpha_n \ge \lceil\frac{q}{2}\rceil \cdot q^{n-1}$ *or* $\alpha_n = \lceil\frac{q}{2}\rceil L$ *for some* $1 \le L < q^{n-1}$ *and there exists a* $\lceil\frac{q}{2}\rceil$*-regular subgraph in* $\mathcal{B}_q(n-1)$ *with* L *vertices, then there exists a fix-free code which fits to* $(\alpha_n)_{n\in\mathbb{N}}$*.*

(vi) **(Binary case : Yekhanin [8], without a full proof.**
General case : This survey, Chapter 4.)
Let $n := \min\{l \,|\, \alpha_l \neq 0\}$*. If* $\frac{\alpha_n}{q^n} \geq \left\lceil \frac{q}{2} \right\rceil \cdot q^{-1}$*, then there exists a fix-free code which fits to* $(\alpha_n)_{n\in\mathbb{N}}$*.*

(vii) **(Binary case : Kukorelly and Zeger [10].)**
Let $|\mathcal{A}| = q = 2$*. If* $\alpha_l \leq 2$ *for all* $l \in \mathbb{N}$ *and* $\sup\{l \,|\, \alpha_l \neq 0\} < \infty$*, then there exists a fix-free code which fits to* $(\alpha_n)_{n\in\mathbb{N}}$*.*

(viii) **(General case : This survey, Chapter 4 only for even q.)**
Let $|\mathcal{A}| = q$ *with* q *even. If there exists an* $n \geq 2$*, with* $\alpha_1 = 0$*,* $\alpha_l = (\frac{q}{2})^l$ *for* $2 \leq l < n$ *and* $\alpha_n \geq q \cdot (\frac{q}{2})^{n-1}$*, then there exists a fix-free Code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

(ix) **(General case : This survey, Chapter 4 only for even q.)**
Let $|\mathcal{A}| = q$ *with* q *even. If there exists an* $n \geq 3$*, with* $\alpha_1 = \alpha_2 = 0$*,* $\alpha_l = 2 \cdot (\frac{q}{2})^l$ *for* $3 \leq l < n$ *and* $\alpha_n \geq 2q(\frac{q}{2})^{n-1}$*, then there exists a fix-free Code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

(x) **(Binary case : This survey, Chapter 5.)**
Let $|\mathcal{A}| = q = 2$*. If there exists an* $n \geq 2$ *such that* $\alpha_2 = \alpha_{2l+1} = 0$ *for all* $l \in \mathbb{N}_0$*,* $\alpha_{2l} = 2^l$ *for all* $2 \leq l < n$*,* $\alpha_{2n} \geq 2^{n+1}$ *and* $\alpha_{2l} \in \mathbb{N}_0$ *for all* $l > n$*, then there exists a fix-free Code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

(xi) **(Binary case : This survey, Chapter 5.)**
Let $|\mathcal{A}| = q = 2$*. If there exists an* $n \geq 3$ *such that* $\alpha_2 = \alpha_4 = \alpha_{2l+1} = 0$ *for all* $l \in \mathbb{N}_0$*,* $\alpha_{2l} = 2^{l+1}$ *for all* $2 \leq l < n$*,* $\alpha_{2n} \geq 2^{n+2}$ *and* $\alpha_{2l} \in \mathbb{N}_0$ *for all* $l > n$*, then there exists a fix-free code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

(xii) **(Binary case : This survey, Chapter 5.)**
Let $|\mathcal{A}| = q = 2$*.*
If there exists an $n \in \mathbb{N}$ *such that* $\alpha_2 = \alpha_4 = \ldots = \alpha_{2n-2} = \alpha_{2l+1} = 0$ *for all* $l \in \mathbb{N}_0$*,* α_{2n} *is even,* $\frac{\alpha_{2n}}{2^{2n}} + \frac{\alpha_{2n+2}}{2^{2n+2}} \geq \frac{1}{2}$ *and there exists a 2-regular subgraph of* $\mathcal{B}_4(n-1)$ *with* $\frac{\alpha_{2n}}{2}$ *vertices, then there exists a fix-free code* $\mathcal{C} \subseteq \mathcal{A}^+$ *which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

(xiii) **(Binary case : This survey, Chapter 5.)**
Let $|\mathcal{A}| = q = 2$*.*
If there exists an $n \in \mathbb{N}$ *such that* $\alpha_2 = \alpha_4 = \ldots = \alpha_{2n-2} = \alpha_{2l+1} = 0$ *for all* $l \in \mathbb{N}_0$ *and* $\frac{\alpha_{2n}}{2^{2n}} \geq \frac{1}{2}$*, then there exists a fix-free code* $\mathcal{C} \subseteq \mathcal{A}^+$ *which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

(xiv) **(Binary case : This survey, Chapter 5.)**
Let $|\mathcal{A}| = q = 2$*,* $l_{min} := \min\{l \,|\, \alpha_l \neq 0\}$ *and* $l_{max} := \sup\{l \,|\, \alpha_l \neq 0\}$*. If*

$l_{max} < \infty$, $4 \leq l_{min}$ *is even,* $\alpha_{2l+1} = 0$ *for all* $l \in \mathbb{N}_0$ *and* $\alpha_{2l} \leq 2^{\frac{l_{min}}{2}-2+l}$ *for all* $2l \neq l_{max}$*, then there exists a fix-free code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

(xv) **(Binary case : Ye and Yeung [7], by computer research.)**
Let $|\mathcal{A}| = q = 2$*. If* $\alpha_l = 0$ *for all* $l \geq 8$*, then there exists a fix-free code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

(xvi) **(Binary case : Yekhanin [8], by computer research.)**
Let $|\mathcal{A}| = q = 2$*. If* $\alpha_l = 0$ *for all* $l \geq 8$*, then there exists a fix-free code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

The next theorem shows results which are related to the binary $\frac{3}{4}$-conjecture.

Theorem 26

(i) **(Binary case : Ye and Yeung [7].)**
Let $|\mathcal{A}| = q = 2$*. If* $\sup\{l \,|\, \alpha_l \neq 0\} < \infty$, $\alpha_1 = 1$ *and* $\sum_{l=1}^{n} \alpha_l \left(\frac{1}{2}\right)^l \leq \frac{5}{8}$*, then there exists a fix-free code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

(ii) **(Binary case : Yekhanin [9].)**
Let $|\mathcal{A}| = q = 2$*. If* $\sum_{l=1}^{n} \alpha_l \left(\frac{1}{2}\right)^l \leq \frac{5}{8}$*, then there exists a fix-free code which fits to* $(\alpha_l)_{l\in\mathbb{N}}$*.*

(iii) **(Binary case : Ye and Yeung [7].)**
Let $|\mathcal{A}| = q = 2$*. Let* $\vec{l_n} = (l_1, \ldots, l_n) \in \mathbb{N}^n$ *be a lengths sequence and* $h(i) := \min\{j \,|\, l_j = l_{i+1}\}$ *for all* $1 \leq i < n$*. If*
$\prod_{i=1}^{n-1} \big(1 - 2 \sum_{1\leq j\leq i} 2^{-l_i} + (i+1-h(i)) \cdot 2^{-l_i+1} + \sum_{\substack{1 \leq j,k \leq h(i)-1 \\ s.t.\ l_j + l_k \leq l_i + 1}} 2^{-l_k - l_k}\big)^+ > 0$*, then there exists a fix-free code which fits to* $\vec{l_n}$*.*

(iv) **(Binary case : Ye and Yeung [7].)**
Let $|\mathcal{A}| = q = 2$*. Let* $\vec{l_n} = (l_1, \ldots, l_n) \in \mathbb{N}^n$ *be a lengths sequence and* $h(i) := \min\{j \,|\, l_j = l_{i+1}\}$ *for all* $1 \leq i < n$*. If*
$\prod_{i=1}^{n-1} \big(1 - 2 \sum_{1\leq j\leq i} 2^{-l_i} + (i+1-h(i)) \cdot 2^{-l_i+1} + \sum_{1\leq j,k\leq h(i)-1} 2^{(l_{i+1}-l_j-l_k)^+ - l_{i+1}}\big)^+ = 0$*, then there doesn't exists a fix-free code which fits to* $\vec{l_n}$*.*

(v) **(Binary case : Ye and Yeung [7].)**
Let $|\mathcal{A}| = q = 2$*. Let* $\vec{l_n} = (l_1, \ldots, l_n) \in \mathbb{N}^n$ *be a lengths sequence and* $h(i) := \min\{j \,|\, l_j = l_{i+1}\}$ *for all* $1 \leq i < n$*. If*

$\sum_{1\le j\le n} 2^{-l_i} < \frac{1}{2} + \frac{n+2-h(n-1)}{2} \cdot 2^{-l_n}$ *, then there exists a fix-free code which fits to* $\vec{l}_n$.

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